Contextual Applications of Differentiation

AP Calculus BC

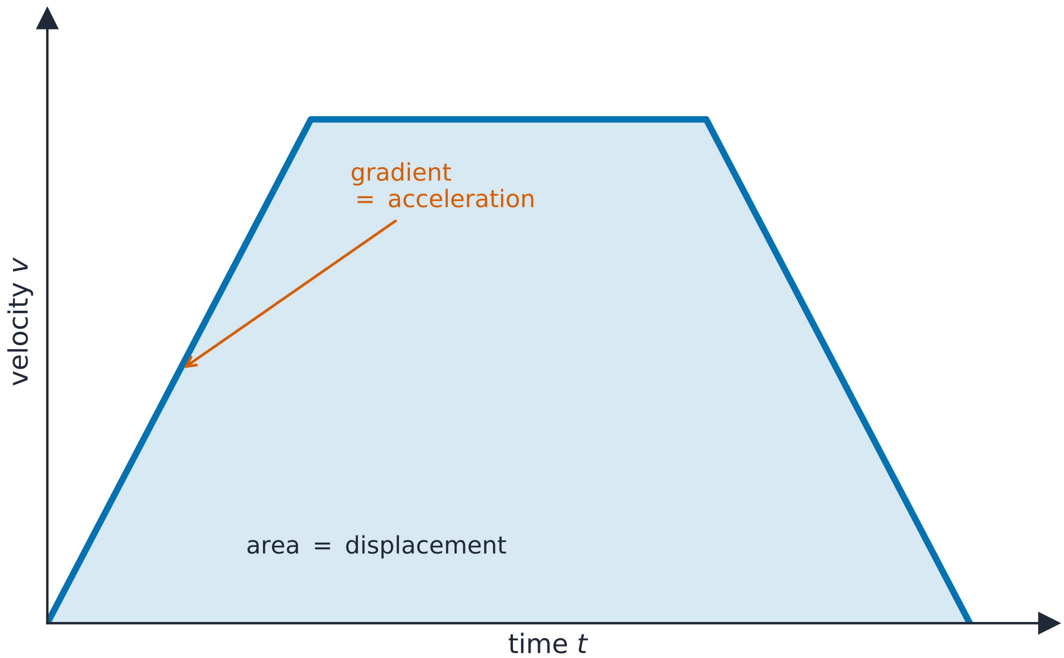
Interpreting the Meaning of the Derivative in Context

Once you can compute derivatives, you use them to describe the real world. The derivative $f'(x)$ **is** the instantaneous rate of change of f with respect to its input. Reading and reporting this rate correctly is a graded skill.

Units matter. The unit of $f'(x)$ is the unit of f divided by the unit of x. If $C(t)$ is a number of acres and t is in weeks, then $C'(t)$ is in **acres per week** 英亩每周. On the exam, "Using correct units, interpret the meaning of $g'(140)$" wants a full sentence: the *value*, the *quantity*, the *rate word* "per", and the *moment*. For example: "$g'(140) = 2.3$ means that at $x = 140$, the quantity is increasing at about 2.3 units per unit of x."

Straight-Line Motion: Position, Velocity, and Acceleration

For a particle moving on a line, three functions of time are linked by differentiation:



On a velocity-time graph, the area is displacement and the gradient is acceleration

- **position** 位置 $s(t)$;
- **velocity** 速度 $v(t) = s'(t)$ –signed; its sign gives direction;
- **acceleration** 加速度 $a(t) = v'(t) = s''(t)$.

Key readings (frequent exam parts):

- The particle is **at rest** 静止 when $v(t) = 0$.
- It moves **right/up** when $v(t) > 0$ and **left/down** when $v(t) < 0$; it **changes direction** where v changes sign.
- **Speed** 速率 is $|v(t)|$. Speed is *increasing* when v and a have the **same sign** (the particle is speeding up), and *decreasing* when they have **opposite signs**.

Distinguish carefully between velocity (has direction) and speed (does not) –the exam tests this exact difference.

Worked example. A particle moves with $s(t) = t^3-6t^2+9t$. Then $v(t) = 3(t-1)(t-3)$, so it is at rest at $t = 1$ and $t = 3$ and changes direction at each. At $t = 2$, $v = -3 < 0$ and $a(2) = 6(2) - 12 = 0$; just after, $a > 0$ while $v < 0$, so the particle is **slowing down** there.

Rates of Change in Applied Contexts Other Than Motion

The same derivative idea models any changing quantity: a draining tank, a spreading population, a cooling cup. Whenever a problem says "the rate at which...", it is describing a derivative. Read the units to know which quantity's rate you have, then interpret in context.

Introduction to Related Rates

In a **related rates** 相关变化率 problem, several quantities change together over time, and you know some rates but want another. The engine is the chain rule: differentiate a relationship **with respect to time** t. Every variable becomes a function of t, so each derivative picks up a "$/\,dt$" factor. Product and quotient rules may also be needed.

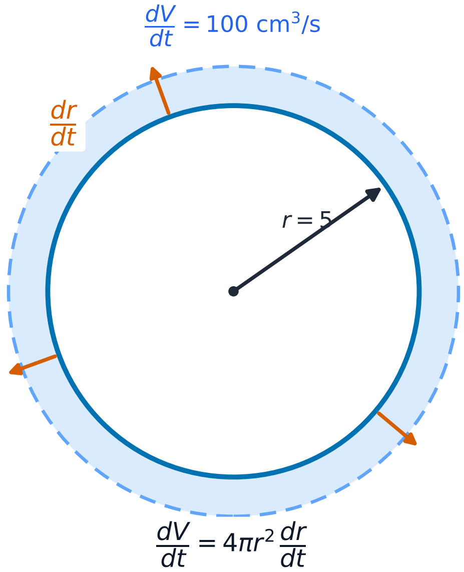
Solving Related Rates Problems

A reliable procedure –and a full-credit template on the exam:

1. **Name the variables** and write down the given rates and the unknown rate (e.g. "$\frac{dh}{dt} = -2$ cm/day, find $\frac{dV}{dt}$").
2. **Write an equation** relating the quantities (often a geometric or volume formula).
3. **Differentiate both sides with respect to** t (chain rule) –*before* substituting numbers.
4. **Substitute** the known values at the instant of interest, and solve for the unknown rate.
5. State the answer with **units** and the correct **sign** (a decreasing quantity has a negative rate).

Substituting numbers too early is the classic error: differentiate the *general* relationship first, then plug in.

Worked example. A spherical balloon's volume grows at $\frac{dV}{dt} = 100$ cm^3/s. From $V = \frac{4}{3}\pi r^3$, differentiate first: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. At $r = 5$, $100 = 4\pi(25)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{\pi} \approx 0.32$ cm/s.



An inflating balloon links dV/dt and dr/dt through the chain rule

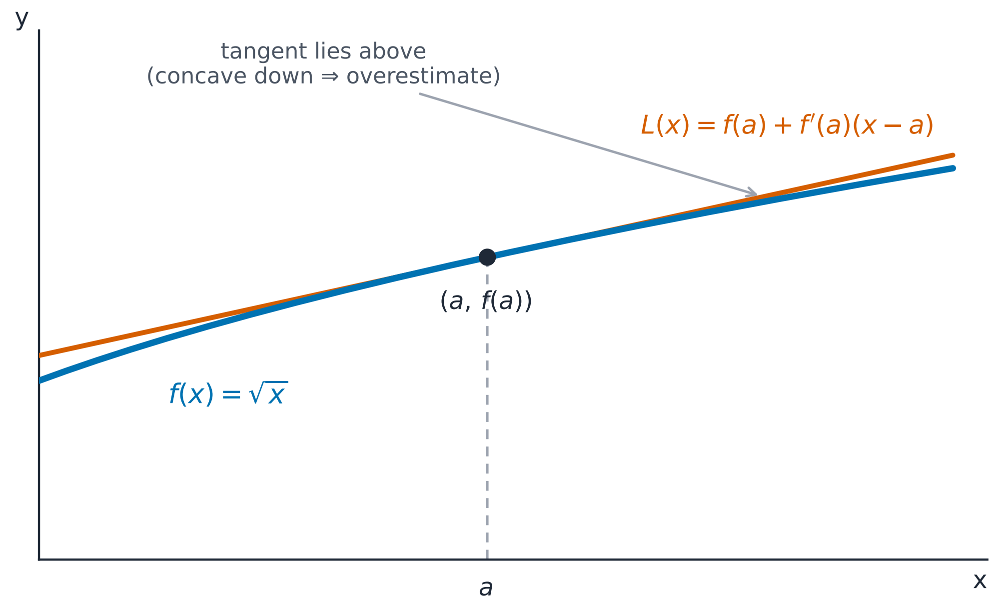
Approximating Values Using Local Linearity and Linearization

Near a point of tangency, a smooth curve looks like its tangent line –this is **local linearity** 局部线性. So the tangent line gives a **linear approximation** 线性近似 (linearization) of the function near that point:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

Use it to estimate f at an x close to a.

Over- or underestimate? The answer follows from **concavity** 凹凸性. If the graph is **concave up** near a (it curves above its tangent), the tangent-line value is an **under-estimate** 低估. If it is **concave down**, the tangent line lies above the curve, giving an **overestimate** 高估. Exam parts test this reasoning, so justify with the sign of f''.



The tangent line is a local linear approximation; concavity fixes over- or under-estimate

Using L'Hospital's Rule for Indeterminate Forms

When direct substitution in a quotient of limits gives the **indeterminate form** 未定式 $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, you may use **L'Hospital's Rule** 洛必达法则:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided the right-hand limit exists. **Differentiate the top and bottom separately** (this is *not* the quotient rule), then try the limit again. First confirm the form really is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ –applying the rule to any other form is a mistake.

Worked example. $\lim_{x \to 0} \dfrac{\sin x}{x}$ gives $\frac{0}{0}$, so differentiate top and bottom: $\lim_{x \to 0} \dfrac{\cos x}{1} = 1$. And $\lim_{x \to 0} \dfrac{e^{2x} - 1}{x}$ is also $\frac{0}{0}$; it becomes $\lim_{x \to 0} \dfrac{2e^{2x}}{1} = 2$.

Exam tips

- In motion problems: velocity is the derivative of position, acceleration the derivative of velocity; **speed increases when velocity and acceleration share a sign**.
- For **related rates**, differentiate the relating equation with respect to **time**, then substitute the given values last.
- Use the tangent line for a **linear approximation** near a known point; it is accurate only close by.
- Read the sign of a rate: positive means the quantities move together, negative means opposite.
- Always state units and interpret the answer in context.