# Differentiation: Composite, Implicit, and Inverse Functions

AP Calculus BC

## The Chain Rule

Unit 2 differentiated single functions. Unit 3 differentiates functions built *inside* other functions. The **chain rule** 链式法则 differentiates a **composite function** 复合函数 $f\big(g(x)\big)$:

$$\frac{d}{dx}\,f\big(g(x)\big) = f'\big(g(x)\big)\cdot g'(x).$$

"Derivative of the outer function (leaving the inside alone), times the derivative of the inside." The inner derivative $g'(x)$ is the piece students forget, so always ask "what is the inside, and what is its derivative?" Example:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\cdot 2x.$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$, the rule reads $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$ –the intermediate $du$ appears to "cancel." Exam questions often give a table for $f$, $g$, $f'$, $g'$ and ask for $h'(a)$ where $h(x) = f\big(g(x)\big)$; evaluate $f'\big(g(a)\big)\cdot g'(a)$ by reading values.

**Worked example.** Differentiate $h(x) = (2x^2+1)^5$. The outer function is "(something)$^5$" and the inner is $2x^2+1$:

$$h'(x) = 5(2x^2+1)^4\cdot 4x = 20x(2x^2+1)^4.$$

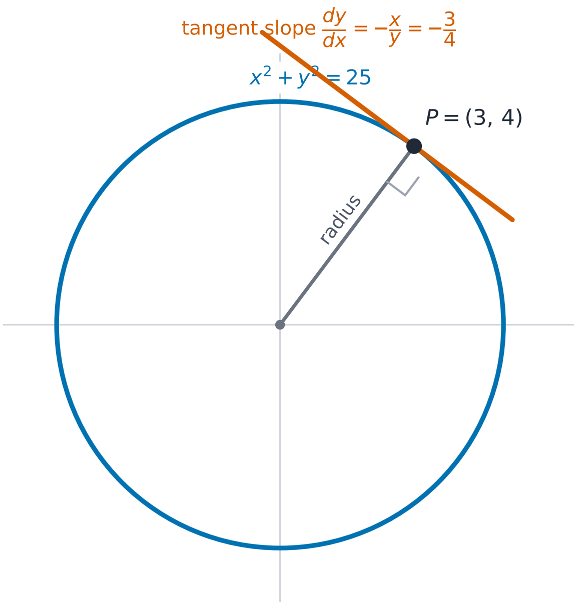
## Implicit Differentiation

Some curves are defined **implicitly** –by an equation in $x$ and $y$ that is not solved for $y$, such as $x^2+y^2=25$. **Implicit differentiation** 隐函数求导 finds $\dfrac{dy}{dx}$ without solving for $y$ first. It is just the chain rule, treating $y$ as a function of $x$.

The method: differentiate **both sides** with respect to $x$; every time you differentiate a $y$-term, multiply by $\dfrac{dy}{dx}$ (the chain rule); then solve algebraically for $\dfrac{dy}{dx}$. For $x^2+y^2=25$:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

**Worked example.** Find the tangent to $x^2+y^2=25$ at $(3,4)$. Here $\dfrac{dy}{dx} = -\dfrac{3}{4}$, so the tangent line is $y-4 = -\frac{3}{4}(x-3)$ –perpendicular to the radius, as geometry predicts.




*Implicit differentiation gives the tangent to a circle, perpendicular to the radius*

**Exam skill –"Show that $\frac{dy}{dx} = \ldots$".** This exact prompt appears most years (e.g. "Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$"). Because the target is given, you must show **every algebra step** cleanly: differentiate both sides, use the product/chain rules on mixed $xy$ terms, collect all $\frac{dy}{dx}$ terms on one side, factor, and divide. A correct final line that skips the algebra earns little. Follow-up parts then ask for a tangent line, or where the tangent is horizontal ($\frac{dy}{dx} = 0$, so the numerator is 0) or vertical (the denominator is 0).

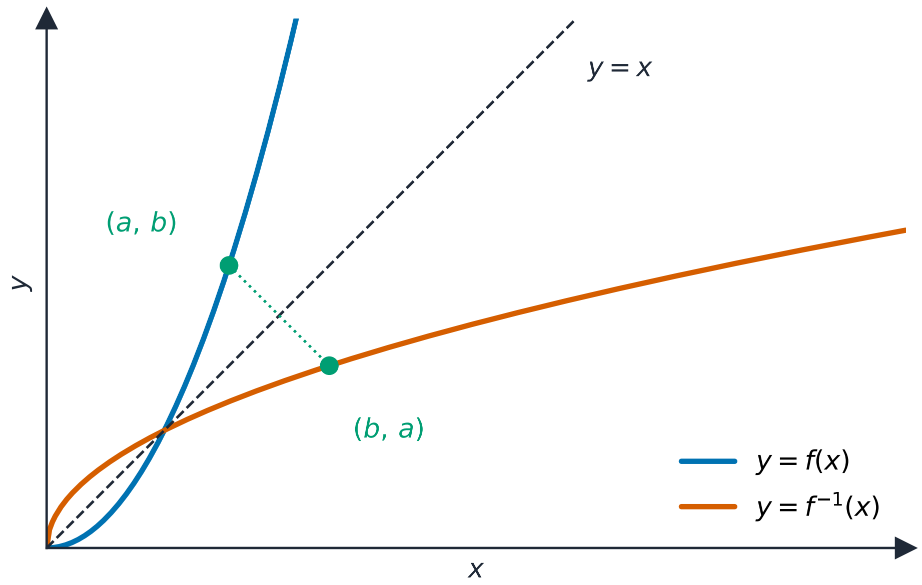
## Differentiating Inverse Functions

If $g$ is the **inverse function** 反函数 of $f$ (so $f(g(x)) = x$), the chain rule links their derivatives:

$$g'(x) = \frac{1}{f'\big(g(x)\big)}, \qquad \text{provided } f'\big(g(x)\big) \neq 0.$$

In words: the derivative of the inverse at a point is the **reciprocal** 倒数 of the derivative of the original function at the *matching* point. A common exam setup gives a table and a point $(a, b)$ on $f$ (so $(b, a)$ is on $g$), then asks for $g'(b) = \frac{1}{f'(a)}$.

**Worked example.** If $f(2) = 5$ and $f'(2) = 3$, and $g$ is the inverse of $f$, then $(5, 2)$ lies on $g$ and $g'(5) = \frac{1}{f'(2)} = \frac{1}{3}$.




*The inverse function is the mirror image of the function in the line $y = x$*

## Differentiating Inverse Trigonometric Functions

The same idea gives the derivatives of the **inverse trigonometric functions** 反三角函数. The three you should know:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

Combine these with the chain rule when the input is itself a function, e.g. $\frac{d}{dx}\arctan(3x) = \frac{3}{1+9x^2}$.

## Selecting Procedures for Calculating Derivatives

A skill topic: real derivatives mix several rules, so read the structure of the expression first, from the outside in.

- Is it a **sum**? Differentiate term by term.
- A **product** or **quotient**? Apply that rule, and expect to use the chain rule inside.
- A **composite** (something inside something)? Chain rule.
- Given **implicitly**? Implicit differentiation.

Name the outermost operation, apply its rule, and recurse inward. Neatness prevents the sign and bookkeeping errors that cost marks.

# Calculating Higher-Order Derivatives

Differentiating $f'$ produces the **second derivative** 二阶导数 $f''$; repeating gives higher-order derivatives. The notations:

$$f''(x) = \frac{d^2y}{dx^2} = y'', \qquad \text{and in general} \qquad f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

The second derivative measures how the *slope* is changing; it drives **concavity** 凹凸性 and acceleration in later units. To find $f''$ implicitly, differentiate the expression for $\frac{dy}{dx}$ again (with the quotient and chain rules), then substitute $\frac{dy}{dx}$ back in.

# Exam tips

- Use the **chain rule** for composite functions —differentiate the outside, then multiply by the derivative of the inside (the most-forgotten factor).
- For **implicit** differentiation, differentiate both sides with respect to $x$ and attach $\frac{dy}{dx}$ each time $y$ is differentiated, then solve.
- Get the **second derivative** by differentiating twice (velocity $\to$ acceleration).
- Combine rules carefully in layered expressions (chain inside product, etc.).
- The inverse function's graph is the reflection in $y = x$; its slope is the reciprocal of the original's at the matching point.