

**Part A (BC): Graphing calculator required****Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the  $xy$ -plane is at position  $(x(t), y(t))$  at time  $t > 0$ . The particle moves in such a way that  $\frac{dx}{dt} = \sqrt{1+t^2}$  and  $\frac{dy}{dt} = \ln(2+t^2)$ . At time  $t = 4$ , the particle is at the point  $(1, 5)$ .

| Model Solution                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | Scoring                      |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------|
| <p>(a) Find the slope of the line tangent to the path of the particle at time <math>t = 4</math>.</p> $\left. \frac{dy}{dx} \right _{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$ <p>The slope of the line tangent to the path of the particle at time <math>t = 4</math> is 0.701.</p>                                                                                                                                                                                                                                                                                                                                  | <p>Answer <b>1 point</b></p> |
| <p><b>Scoring notes:</b></p> <ul style="list-style-type: none"> <li>To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point: <math>\frac{y'(4)}{x'(4)} = 0.701</math>, <math>\frac{\ln(2+4^2)}{\sqrt{1+4^2}}</math>, or <math>\frac{\ln 18}{\sqrt{17}}</math>.</li> <li>Note: A response with an incorrect equation of the form “function = constant”, such as <math>\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}</math>, will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.</li> </ul> |                              |
| <p style="text-align: right;"><b>Total for part (a)      1 point</b></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                              |

- (b) Find the speed of the particle at time  $t = 4$ , and find the acceleration vector of the particle at time  $t = 4$ .

|                                                                                                                                                                                                                                                                    |                                  |         |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------|---------|
| $\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$ <p>The speed of the particle at time <math>t = 4</math> is 5.035.</p>                                                                                                                           | Speed                            | 1 point |
| $a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$ <p>The acceleration vector of the particle at time <math>t = 4</math> is <math>\langle 0.970, 0.444 \rangle</math>.</p> | First component of acceleration  | 1 point |
|                                                                                                                                                                                                                                                                    | Second component of acceleration | 1 point |

**Scoring notes:**

- To earn any of these points, the setup used to perform the calculation must be evident in the response. For example,  $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$  or  $\sqrt{17 + (\ln 18)^2}$  earns the first point, and  $\langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$  earns both the second and third points.
- The second and third points can be earned independently.
- If the acceleration vector is not presented as an ordered pair, the  $x$ - and  $y$ -components must be labeled.
- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.
- A response which correctly calculates expressions for both  $x''(t) = \frac{t}{\sqrt{1+t^2}}$  and  $y''(t) = \frac{2t}{2+t^2}$ , but which fails to evaluate both of these expressions at  $t = 4$ , earns only 1 of the last 2 points.
- An unsupported acceleration vector earns only 1 of the last 2 points.

**Total for part (b) 3 points**

- (c) Find the  $y$ -coordinate of the particle's position at time  $t = 6$ .

|                                                                                       |             |         |
|---------------------------------------------------------------------------------------|-------------|---------|
| $y(6) = y(4) + \int_4^6 \ln(2 + t^2) dt$                                              | Integrand   | 1 point |
|                                                                                       | Uses $y(4)$ | 1 point |
| $= 5 + 6.570517 = 11.570517$                                                          | Answer      | 1 point |
| The $y$ -coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570). |             |         |

**Scoring notes:**

- For the first point, an integrand of  $\ln(2 + t^2)$  can appear in either an indefinite integral or an incorrect definite integral.
- A definite integral with incorrect limits is not eligible for the answer point.
- Similarly, an indefinite integral is not eligible for the answer point.
- For the second point, the value for  $y(4)$  must be added to a definite integral.
- A response that reports the correct  $x$ -coordinate of the particle's position at time  $t = 6$  as  $x(6) = x(4) + \int_4^6 \sqrt{1 + t^2} dt = 11.200$  (or 11.201) instead of the  $y$ -coordinate, earns 2 out of the 3 points.
- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).
- If the differential is missing:
  - $y(6) = \int_4^6 \ln(2 + t^2)$  earns the first point and is eligible for the third.
  - $y(6) = \int_4^6 \ln(2 + t^2) + y(4)$  does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
  - $y(6) = y(4) + \int_4^6 \ln(2 + t^2)$  earns the first two points and is eligible for the third.

**Total for part (c)     3 points**

- (d) Find the total distance the particle travels along the curve from time  $t = 4$  to time  $t = 6$ .

|                                                                                                      |           |         |
|------------------------------------------------------------------------------------------------------|-----------|---------|
| $\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$                     | Integrand | 1 point |
| $= 12.136228$                                                                                        | Answer    | 1 point |
| The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136. |           |         |

**Scoring notes:**

- The first point is earned for presenting the correct integrand in a definite integral.
- To earn the second point, a response must have earned the first point and must present the value 12.136.
- An unsupported answer of 12.136 does not earn either point.

**Total for part (d) 2 points**

**Total for question 2 9 points**

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# AP<sup>®</sup> Calculus BC

## 2015 Scoring Guidelines

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# 2015 SCORING GUIDELINES

## Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function  $R$ , where  $R(t) = 20\sin\left(\frac{t^2}{35}\right)$  cubic feet per hour,  $t$  is measured in hours, and  $0 \leq t \leq 8$ . The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by  $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$  cubic feet per hour, for  $0 \leq t \leq 8$ . There are 30 cubic feet of water in the pipe at time  $t = 0$ .

- How many cubic feet of rainwater flow into the pipe during the 8-hour time interval  $0 \leq t \leq 8$ ?
- Is the amount of water in the pipe increasing or decreasing at time  $t = 3$  hours? Give a reason for your answer.
- At what time  $t$ ,  $0 \leq t \leq 8$ , is the amount of water in the pipe at a minimum? Justify your answer.
- The pipe can hold 50 cubic feet of water before overflowing. For  $t > 8$ , water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time  $w$  when the pipe will begin to overflow.

(a)  $\int_0^8 R(t) dt = 76.570$

2 :  $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b)  $R(3) - D(3) = -0.313632 < 0$   
 Since  $R(3) < D(3)$ , the amount of water in the pipe is decreasing at time  $t = 3$  hours.

2 :  $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time  $t$ ,  $0 \leq t \leq 8$ , is  $30 + \int_0^t [R(x) - D(x)] dx$ .

3 :  $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$

| $t$      | Amount of water in the pipe |
|----------|-----------------------------|
| 0        | 30                          |
| 3.271658 | 27.964561                   |
| 8        | 48.543686                   |

The amount of water in the pipe is a minimum at time  $t = 3.272$  (or 3.271) hours.

(d)  $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

# 2015 SCORING GUIDELINES

## Question 2

At time  $t \geq 0$ , a particle moving along a curve in the  $xy$ -plane has position  $(x(t), y(t))$  with velocity vector  $v(t) = (\cos(t^2), e^{0.5t})$ . At  $t = 1$ , the particle is at the point  $(3, 5)$ .

- (a) Find the  $x$ -coordinate of the position of the particle at time  $t = 2$ .
- (b) For  $0 < t < 1$ , there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time  $t = 0$  to time  $t = 1$ .

(a)  $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$  (or 2.556)

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$$

(b)  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

$$2 : \begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$$

(c) Speed =  $\sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

$$2 : \begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$$

(d) Distance =  $\int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$  (or 1.594)

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

## 2015 SCORING GUIDELINES

## Question 3

|                               |   |     |     |      |     |
|-------------------------------|---|-----|-----|------|-----|
| $t$<br>(minutes)              | 0 | 12  | 20  | 24   | 40  |
| $v(t)$<br>(meters per minute) | 0 | 200 | 240 | -220 | 150 |

Johanna jogs along a straight path. For  $0 \leq t \leq 40$ , Johanna's velocity is given by a differentiable function  $v$ . Selected values of  $v(t)$ , where  $t$  is measured in minutes and  $v(t)$  is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of  $v'(16)$ .
- (b) Using correct units, explain the meaning of the definite integral  $\int_0^{40} |v(t)| dt$  in the context of the problem.

Approximate the value of  $\int_0^{40} |v(t)| dt$  using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For  $0 \leq t \leq 10$ , Bob's velocity is modeled by  $B(t) = t^3 - 6t^2 + 300$ , where  $t$  is measured in minutes and  $B(t)$  is measured in meters per minute.

Find Bob's acceleration at time  $t = 5$ .

- (d) Based on the model  $B$  from part (c), find Bob's average velocity during the interval  $0 \leq t \leq 10$ .

(a)  $v'(16) \approx \frac{240 - 200}{20 - 12} = 5 \text{ meters/min}^2$

1 : approximation

- (b)  $\int_0^{40} |v(t)| dt$  is the total distance Johanna jogs, in meters, over the time interval  $0 \leq t \leq 40$  minutes.

3 :  $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is  $B'(t) = 3t^2 - 12t$ .  
 $B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$

2 :  $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

(d) Avg vel  $= \frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$\begin{aligned} &= \frac{1}{10} \left[ \frac{t^4}{4} - 2t^3 + 300t \right]_0^{10} \\ &= \frac{1}{10} \left[ \frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min} \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

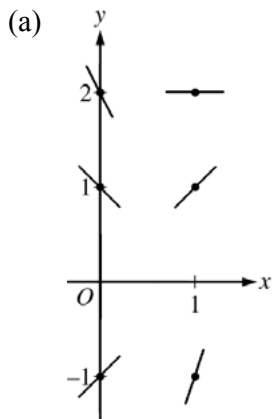


## 2015 SCORING GUIDELINES

## Question 4

Consider the differential equation  $\frac{dy}{dx} = 2x - y$ .

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let  $y = f(x)$  be the particular solution to the differential equation with the initial condition  $f(2) = 3$ . Does  $f$  have a relative minimum, a relative maximum, or neither at  $x = 2$ ? Justify your answer.
- (d) Find the values of the constants  $m$  and  $b$  for which  $y = mx + b$  is a solution to the differential equation.



$$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$$

(b)  $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II,  $x < 0$  and  $y > 0$ , so  $2 - 2x + y > 0$ .  
Therefore, all solution curves are concave up in Quadrant II.

$$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$$

(c)  $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore,  $f$  has neither a relative minimum nor a relative maximum at  $x = 2$ .

$$2 : \begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$$

(d)  $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

$$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$$

## 2015 SCORING GUIDELINES

## Question 5

Consider the function  $f(x) = \frac{1}{x^2 - kx}$ , where  $k$  is a nonzero constant. The derivative of  $f$  is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let  $k = 3$ , so that  $f(x) = \frac{1}{x^2 - 3x}$ . Write an equation for the line tangent to the graph of  $f$  at the point whose  $x$ -coordinate is 4.
- (b) Let  $k = 4$ , so that  $f(x) = \frac{1}{x^2 - 4x}$ . Determine whether  $f$  has a relative minimum, a relative maximum, or neither at  $x = 2$ . Justify your answer.
- (c) Find the value of  $k$  for which  $f$  has a critical point at  $x = -5$ .
- (d) Let  $k = 6$ , so that  $f(x) = \frac{1}{x^2 - 6x}$ . Find the partial fraction decomposition for the function  $f$ .

Find  $\int f(x) dx$ .

$$(a) \quad f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4} \quad f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$$

An equation for the line tangent to the graph of  $f$  at the point whose  $x$ -coordinate is 4 is  $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$ .

$$(b) \quad f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2} \quad f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$$

$f'(x)$  changes sign from positive to negative at  $x = 2$ .  
Therefore,  $f$  has a relative maximum at  $x = 2$ .

$$(c) \quad f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$$

$$(d) \quad \frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left( \frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

2 :  $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

2 :  $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

1 : answer

4 :  $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 6

The Maclaurin series for a function  $f$  is given by  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$  and converges to  $f(x)$  for  $|x| < R$ , where  $R$  is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find  $R$ .
- (b) Write the first four nonzero terms of the Maclaurin series for  $f'$ , the derivative of  $f$ . Express  $f'$  as a rational function for  $|x| < R$ .
- (c) Write the first four nonzero terms of the Maclaurin series for  $e^x$ . Use the Maclaurin series for  $e^x$  to write the third-degree Taylor polynomial for  $g(x) = e^x f(x)$  about  $x = 0$ .

- (a) Let  $a_n$  be the  $n$ th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is  $R = \frac{1}{3}$ .

- (b) The first four nonzero terms of the Maclaurin series for  $f'$  are  $1 - 3x + 9x^2 - 27x^3$ .

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for  $e^x$  are

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The product of the Maclaurin series for  $e^x$  and the Maclaurin series for  $f$  is

$$\begin{aligned} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \left( x - \frac{3}{2}x^2 + 3x^3 - \cdots \right) \\ = x - \frac{1}{2}x^2 + 2x^3 + \cdots \end{aligned}$$

The third-degree Taylor polynomial for  $g(x) = e^x f(x)$

about  $x = 0$  is  $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$ .

3 :  $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

3 :  $\begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$

3 :  $\begin{cases} 1 : \text{first four nonzero terms} \\ \quad \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$