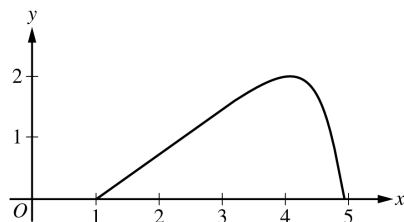


**Part A (BC): Graphing calculator required****Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



For  $0 \leq t \leq \pi$ , a particle is moving along the curve shown so that its position at time  $t$  is  $(x(t), y(t))$ , where  $x(t)$  is not explicitly given and  $y(t) = 2\sin t$ . It is known that  $\frac{dx}{dt} = e^{\cos t}$ . At time  $t = 0$ , the particle is at position  $(1, 0)$ .

**Model Solution****Scoring**

- (a) Find the acceleration vector of the particle at time  $t = 1$ . Show the setup for your calculations.

$$x''(1) = \left. \frac{d}{dt}(e^{\cos t}) \right|_{t=1} = -1.444407$$

$x''(1)$  with setup **1 point**

$$y(t) = 2\sin t \Rightarrow y'(t) = 2\cos t$$

$$y''(1) = \left. \frac{d}{dt}(2\cos t) \right|_{t=1} = -1.682942$$

$y''(1)$  with setup **1 point**

The acceleration vector at time  $t = 1$  is

$$a(1) = \langle -1.444, -1.683 \text{ (or } -1.682) \rangle.$$

**Scoring notes:**

- The exact answer is  $\langle x''(1), y''(1) \rangle = \langle -e^{\cos 1} \sin 1, -2\sin 1 \rangle$ .
- $\langle -e^{\cos t} \sin t, -2\sin t \rangle$  together with an incorrect or missing evaluation at  $t = 1$  earns 1 of the 2 points.

- A response of  $\langle -e^{\cos t} \sin t, -2\sin t \rangle = \langle -e^{\cos 1} \sin 1, -2\sin 1 \rangle$  or equivalent earns only 1 of the 2 points because it equates an expression to a numerical value.
- An unsupported correct acceleration vector earns 1 of the 2 points.
- The acceleration vector may be presented with other symbols, for example  $(\ , \ )$  or  $[ \ , \ ]$ .
- The components may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode,  $x''(1) = -0.000828$  or  $-0.047433$  and  $y''(1) = -0.000609$  or  $-0.034905$ . A response that presents one of these values with correct setups earns 1 of the 2 points.

**Total for part (a) 2 points**

- (b) For  $0 \leq t \leq \pi$ , find the first time  $t$  at which the speed of the particle is 1.5. Show the work that leads to your answer.

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$$

$$\sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5 \quad \text{1 point}$$

$$0 \leq t \leq \pi \text{ and } \sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5$$

$$\Rightarrow t = 1.254472, t = 2.358077$$

Answer **1 point**

The first time at which the speed of the particle is 1.5 is  $t = 1.254$ .

**Scoring notes:**

- A response with an implied equation is eligible for both points. For example, a response of “Speed =  $\sqrt{(e^{\cos t})^2 + (2\cos t)^2}$  and is first equal to 1.5 at  $t = 1.254$ ” earns both points.
- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5$  earns the first point. Speed = 1.5 by itself does not earn the first point. Both of these responses are eligible to earn the second point.
- A response need not consider the value  $t = 2.358077$ .
- A response of  $t = 1.254$  alone does not earn either point.
- A response with a parenthesis error(s) in either  $(e^{\cos t})^2$  or  $(2\cos t)^2$  does not earn the first point but does earn the second point for the correct answer. Note:  $\sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$  is not considered a parenthesis error.

- Degree mode: In degree mode,  $\sqrt{(e^{\cos t})^2 + (2 \cos t)^2} = 1.5$  has no solution for  $0 \leq t \leq \pi$ .

A response that finds no time  $t$  at which the speed of the particle is 1.5 cannot be assumed to be working in degree mode.

**Total for part (b) 2 points**

- (c) Find the slope of the line tangent to the path of the particle at time  $t = 1$ . Find the  $x$ -coordinate of the position of the particle at time  $t = 1$ . Show the work that leads to your answers.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{e^{\cos t}}$$

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{2 \cos 1}{e^{\cos 1}} = 0.629530$$

The slope of the line tangent to the curve at  $t = 1$  is 0.630 (or 0.629).

$$x(1) = x(0) + \int_0^1 \frac{dx}{dt} dt = 1 + \int_0^1 e^{\cos t} dt = 3.341575$$

The  $x$ -coordinate of the position at  $t = 1$  is 3.342 (or 3.341).

Slope with supporting work **1 point**

$$\int_0^1 e^{\cos t} dt$$

$x(1)$  **1 point**

**Scoring notes:**

- To earn the first point, the response must communicate  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ ; for example:
  - $\frac{dy}{dx} = \frac{2 \cos 1}{e^{\cos 1}}$
  - $\frac{dy/dt}{dx/dt} = 0.63$
  - $x'(1) = 1.716526$ ,  $y'(1) = 1.080605$ , slope = 0.63
  - $\frac{dy}{dt} = 2 \cos t$ , slope = 0.63
- A response may import an incorrect expression for  $y'(t)$  or value of  $y'(1)$  from part (a), provided it was declared in part (a).
- The second point is earned for a response that presents the definite integral  $\int_0^1 e^{\cos t} dt$  or  $\int_0^1 \frac{dx}{dt} dt$  with or without the initial condition.

- For the second point, if the differential is missing:

- $\int_0^1 e^{\cos t} dt$  earns the second point and is eligible for the third point.

- $x(1) = \int_0^1 e^{\cos t} dt$  earns the second point but is not eligible for the third point.

- $x(1) = 1 + \int_0^1 e^{\cos t} dt$  earns the second point and is eligible for the third point.

- $x(1) = \int_0^1 e^{\cos t} + 1 dt$  does not earn the second point but earns the third point for the correct answer.

- The third point is not earned for a response that presents an incorrect statement, such as

$$x(1) = \int_0^1 e^{\cos t} dt = 1 + 2.342.$$

- Degree mode: In degree mode,  $\frac{dy}{dx} = 0.735759$  or 0.012841 and  $1 + \int_0^1 e^{\cos t} dt = 3.718144$ .

**Total for part (c) 3 points**

- (d) Find the total distance traveled by the particle over the time interval  $0 \leq t \leq \pi$ . Show the setup for your calculations.

$$\int_0^\pi \sqrt{(e^{\cos t})^2 + (2 \cos t)^2} dt$$

$$= 6.034611$$

The total distance traveled by the particle over  $0 \leq t \leq \pi$  is 6.035 (or 6.034).

Integral **1 point**

Answer **1 point**

**Scoring notes:**

- The first point is earned for presenting the correct integrand in a definite integral.
- Parentheses errors were assessed in part (b) and, therefore, will not affect the scoring in part (d).
- If the integrand is an incorrect speed function imported from part (b), the response earns the first point and does not earn the second point.
- An unsupported answer of 6.035 (or 6.034) does not earn either point.
- Degree mode: In degree mode, the total distance is 10.596835 or 8.536161.

**Total for part (d) 2 points**

**Total for question 2 9 points**

**Part A (BC): Graphing calculator required****Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the  $xy$ -plane is at position  $(x(t), y(t))$  at time  $t > 0$ . The particle moves in such a way that  $\frac{dx}{dt} = \sqrt{1+t^2}$  and  $\frac{dy}{dt} = \ln(2+t^2)$ . At time  $t = 4$ , the particle is at the point  $(1, 5)$ .

**Model Solution****Scoring**

- (a) Find the slope of the line tangent to the path of the particle at time  $t = 4$ .

$$\left. \frac{dy}{dx} \right|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$$

Answer

**1 point**

The slope of the line tangent to the path of the particle at time  $t = 4$  is 0.701.

**Scoring notes:**

- To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point:  $\frac{y'(4)}{x'(4)} = 0.701$ ,  $\frac{\ln(2+4^2)}{\sqrt{1+4^2}}$ , or  $\frac{\ln 18}{\sqrt{17}}$ .
- Note: A response with an incorrect equation of the form “function = constant”, such as  $\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}$ , will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.

**Total for part (a) 1 point**

- (b) Find the speed of the particle at time  $t = 4$ , and find the acceleration vector of the particle at time  $t = 4$ .

$$\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$$

Speed

**1 point**

The speed of the particle at time  $t = 4$  is 5.035.

$$a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$$

First component of acceleration

**1 point**

The acceleration vector of the particle at time  $t = 4$  is  $\langle 0.970, 0.444 \rangle$ .

Second component of acceleration

**1 point****Scoring notes:**

- To earn any of these points, the setup used to perform the calculation must be evident in the response. For example,  $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$  or  $\sqrt{17 + (\ln 18)^2}$  earns the first point, and  $\langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$  earns both the second and third points.
- The second and third points can be earned independently.
- If the acceleration vector is not presented as an ordered pair, the  $x$ - and  $y$ -components must be labeled.
- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.
- A response which correctly calculates expressions for both  $x''(t) = \frac{t}{\sqrt{1+t^2}}$  and  $y''(t) = \frac{2t}{2+t^2}$ , but which fails to evaluate both of these expressions at  $t = 4$ , earns only 1 of the last 2 points.
- An unsupported acceleration vector earns only 1 of the last 2 points.

**Total for part (b) 3 points**

(c) Find the  $y$ -coordinate of the particle's position at time  $t = 6$ .

|                                                                                       |             |         |
|---------------------------------------------------------------------------------------|-------------|---------|
| $y(6) = y(4) + \int_4^6 \ln(2 + t^2) dt$                                              | Integrand   | 1 point |
|                                                                                       | Uses $y(4)$ | 1 point |
| $= 5 + 6.570517 = 11.570517$                                                          | Answer      | 1 point |
| The $y$ -coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570). |             |         |

**Scoring notes:**

- For the first point, an integrand of  $\ln(2 + t^2)$  can appear in either an indefinite integral or an incorrect definite integral.
- A definite integral with incorrect limits is not eligible for the answer point.
- Similarly, an indefinite integral is not eligible for the answer point.
- For the second point, the value for  $y(4)$  must be added to a definite integral.
- A response that reports the correct  $x$ -coordinate of the particle's position at time  $t = 6$  as  $x(6) = x(4) + \int_4^6 \sqrt{1 + t^2} dt = 11.200$  (or 11.201) instead of the  $y$ -coordinate, earns 2 out of the 3 points.
- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).
- If the differential is missing:
  - $y(6) = \int_4^6 \ln(2 + t^2)$  earns the first point and is eligible for the third.
  - $y(6) = \int_4^6 \ln(2 + t^2) + y(4)$  does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
  - $y(6) = y(4) + \int_4^6 \ln(2 + t^2)$  earns the first two points and is eligible for the third.

**Total for part (c) 3 points**(d) Find the total distance the particle travels along the curve from time  $t = 4$  to time  $t = 6$ .

|                                                                                                      |           |         |
|------------------------------------------------------------------------------------------------------|-----------|---------|
| $\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$                     | Integrand | 1 point |
| $= 12.136228$                                                                                        | Answer    | 1 point |
| The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136. |           |         |

**Scoring notes:**

- The first point is earned for presenting the correct integrand in a definite integral.
- To earn the second point, a response must have earned the first point and must present the value 12.136.
- An unsupported answer of 12.136 does not earn either point.

**Total for part (d) 2 points****Total for question 2 9 points**

**Part A (BC): Graphing calculator required****Question 2****9 points****General Scoring Notes**

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

For time  $t \geq 0$ , a particle moves in the  $xy$ -plane with position  $(x(t), y(t))$  and velocity vector  $\langle (t-1)e^2, \sin(t^{1.25}) \rangle$ . At time  $t = 0$ , the position of the particle is  $(-2, 5)$ .

| Model Solution                                                                                                                                                                                                                                                            | Scoring                                   |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------|
| <p>(a) Find the speed of the particle at time <math>t = 1.2</math>. Find the acceleration vector of the particle at time <math>t = 1.2</math>.</p> $\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271488$ <p>At time <math>t = 1.2</math>, the speed of the particle is 1.271.</p> | <p>Speed <b>1 point</b></p>               |
| $\langle x''(1.2), y''(1.2) \rangle = \langle 6.246630, 0.405125 \rangle$ <p>At time <math>t = 1.2</math>, the acceleration vector of the particle is <math>\langle 6.247</math> (or 6.246), 0.405.</p>                                                                   | <p>Acceleration vector <b>1 point</b></p> |

**Scoring notes:**

- Unsupported answers do not earn any points in this part.
- The acceleration vector may be presented with other symbols, for example  $(\ , \ )$  or  $[ \ , \ ]$ , or the coordinates may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, speed = 0.844 and  $y''(1.2) = 0.023$  (or 0.022).

**Total for part (a) 2 points**

- (b) Find the total distance traveled by the particle over the time interval  $0 \leq t \leq 1.2$ .

|                                                                                                             |           |                |
|-------------------------------------------------------------------------------------------------------------|-----------|----------------|
| $\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2} dt = 1.009817$                                                   | Integrand | <b>1 point</b> |
| The total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$ is 1.010 (or 1.009). | Answer    | <b>1 point</b> |

**Scoring notes:**

- The first point is earned by presenting the integrand  $\sqrt{(x'(t))^2 + (y'(t))^2}$  in a definite integral with any limits. A definite integral with incorrect limits is not eligible for the second point.
- Once earned, the first point cannot be lost. Even in the presence of subsequent copy errors, the correct answer will earn the second point.
- If the first point is not earned because of a copy error, the second point is still earned for a correct answer.
- Unsupported answers will not earn either point.
- Degree mode: distance = 0.677 (or 0.676). (See degree mode statement in part (a).)

**Total for part (b) 2 points**

- (c) Find the coordinates of the point at which the particle is farthest to the left for  $t \geq 0$ . Explain why there is no point at which the particle is farthest to the right for  $t \geq 0$ .

|                                                                                                                                                                                                                                                                                                                       |                                       |         |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------|---------|
| $x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$                                                                                                                                                                                                                                                                          | Sets $x'(t) = 0$                      | 1 point |
| Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$ , the particle is farthest to the left at time $t = 1$ .                                                                                                                                                                                              | Explains leftmost position at $t = 1$ | 1 point |
| $x(1) = -2 + \int_0^1 x'(t) dt = -2.603511$                                                                                                                                                                                                                                                                           | One coordinate of leftmost position   | 1 point |
| $y(1) = 5 + \int_0^1 y'(t) dt = 5.410486$                                                                                                                                                                                                                                                                             |                                       |         |
| The particle is farthest to the left at point $(-2.604 \text{ (or } -2.603), 5.410)$ .                                                                                                                                                                                                                                | Leftmost position                     | 1 point |
| Because $x'(t) > 0$ for $t > 1$ , the particle moves to the right for $t > 1$ .<br><br>Also, $x(2) = -2 + \int_0^2 x'(t) dt > -2 = x(0)$ , so the particle's motion extends to the right of its initial position after time $t = 1$ .<br>Therefore, there is no point at which the particle is farthest to the right. | Explanation                           | 1 point |

**Scoring notes:**

- The second point is earned for presenting a valid reason why the particle is at its leftmost position at time  $t = 1$ . For example, a response could present the argument shown in the model solution, or it could indicate that the only critical point of  $x(t)$  occurs at  $t = 1$  and  $x'(t)$  changes from negative to positive at this time.
- Unsupported positions  $x(1)$  and/or  $y(1)$  do not earn the third (or fourth) point(s).
- Writing  $x(1) = \int_0^1 x'(t) - 2 = -2.603511$  does not earn the third (or fourth) point, because the missing  $dt$  makes this statement unclear or false. However,  $x(1) = -2 + \int_0^1 x'(t) = -2.603511$  does earn the third point, because it is not ambiguous. Similarly, for  $y(1)$ .
- For the fourth point the coordinates of the leftmost point do not have to be written as an ordered pair as long as they are labeled as the  $x$ - and  $y$ -coordinates.
- To earn the last point a response must verify that the particle moves to the right of its initial position (as well as moves to the right for all  $t > 1$ ). Note that there are several ways to demonstrate this.
- Degree mode:  $y$ -coordinate = 5.008 (or 5.007). (See degree mode statement in part (a).)

**Total for part (c) 5 points**

**Total for question 2 9 points**

## AP<sup>®</sup> Calculus BC 2015 Scoring Guidelines

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## 2015 SCORING GUIDELINES

## Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function  $R$ , where  $R(t) = 20\sin\left(\frac{t^2}{35}\right)$  cubic feet per hour,  $t$  is measured in hours, and  $0 \leq t \leq 8$ . The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by  $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$  cubic feet per hour, for  $0 \leq t \leq 8$ . There are 30 cubic feet of water in the pipe at time  $t = 0$ .

- (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval  $0 \leq t \leq 8$ ?
- (b) Is the amount of water in the pipe increasing or decreasing at time  $t = 3$  hours? Give a reason for your answer.
- (c) At what time  $t$ ,  $0 \leq t \leq 8$ , is the amount of water in the pipe at a minimum? Justify your answer.
- (d) The pipe can hold 50 cubic feet of water before overflowing. For  $t > 8$ , water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time  $w$  when the pipe will begin to overflow.

(a)  $\int_0^8 R(t) dt = 76.570$

2 :  $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b)  $R(3) - D(3) = -0.313632 < 0$   
 Since  $R(3) < D(3)$ , the amount of water in the pipe is decreasing at time  $t = 3$  hours.

2 :  $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time  $t$ ,  $0 \leq t \leq 8$ , is  $30 + \int_0^t [R(x) - D(x)] dx$ .

3 :  $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$$

| $t$      | Amount of water in the pipe |
|----------|-----------------------------|
| 0        | 30                          |
| 3.271658 | 27.964561                   |
| 8        | 48.543686                   |

The amount of water in the pipe is a minimum at time  $t = 3.272$  (or 3.271) hours.

(d)  $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 2

At time  $t \geq 0$ , a particle moving along a curve in the  $xy$ -plane has position  $(x(t), y(t))$  with velocity vector  $\mathbf{v}(t) = (\cos(t^2), e^{0.5t})$ . At  $t = 1$ , the particle is at the point  $(3, 5)$ .

- (a) Find the  $x$ -coordinate of the position of the particle at time  $t = 2$ .
- (b) For  $0 < t < 1$ , there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time  $t = 0$  to time  $t = 1$ .

(a)  $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$  (or 2.556)

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b)  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

2 :  $\begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$

(c) Speed =  $\sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

2 :  $\begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$

(d) Distance =  $\int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$  (or 1.594)

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 3

|                               |   |     |     |      |     |
|-------------------------------|---|-----|-----|------|-----|
| $t$<br>(minutes)              | 0 | 12  | 20  | 24   | 40  |
| $v(t)$<br>(meters per minute) | 0 | 200 | 240 | -220 | 150 |

Johanna jogs along a straight path. For  $0 \leq t \leq 40$ , Johanna's velocity is given by a differentiable function  $v$ . Selected values of  $v(t)$ , where  $t$  is measured in minutes and  $v(t)$  is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of  $v'(16)$ .
- (b) Using correct units, explain the meaning of the definite integral  $\int_0^{40} |v(t)| dt$  in the context of the problem.

Approximate the value of  $\int_0^{40} |v(t)| dt$  using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For  $0 \leq t \leq 10$ , Bob's velocity is modeled by  $B(t) = t^3 - 6t^2 + 300$ , where  $t$  is measured in minutes and  $B(t)$  is measured in meters per minute.

Find Bob's acceleration at time  $t = 5$ .

- (d) Based on the model  $B$  from part (c), find Bob's average velocity during the interval  $0 \leq t \leq 10$ .

(a)  $v'(16) \approx \frac{240 - 200}{20 - 12} = 5$  meters/min<sup>2</sup>

1 : approximation

- (b)  $\int_0^{40} |v(t)| dt$  is the total distance Johanna jogs, in meters, over the time interval  $0 \leq t \leq 40$  minutes.

3 :  $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is  $B'(t) = 3t^2 - 12t$ .

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 :  $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

- (d) Avg vel =  $\frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$\begin{aligned} &= \frac{1}{10} \left[ \frac{t^4}{4} - 2t^3 + 300t \right]_0^{10} \\ &= \frac{1}{10} \left[ \frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min} \end{aligned}$$

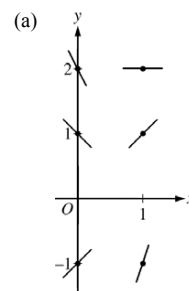
3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 4

Consider the differential equation  $\frac{dy}{dx} = 2x - y$ .

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let  $y = f(x)$  be the particular solution to the differential equation with the initial condition  $f(2) = 3$ . Does  $f$  have a relative minimum, a relative maximum, or neither at  $x = 2$ ? Justify your answer.
- (d) Find the values of the constants  $m$  and  $b$  for which  $y = mx + b$  is a solution to the differential equation.



2 :  $\begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b)  $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II,  $x < 0$  and  $y > 0$ , so  $2 - 2x + y > 0$ .

Therefore, all solution curves are concave up in Quadrant II.

2 :  $\begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c)  $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore,  $f$  has neither a relative minimum nor a relative maximum at  $x = 2$ .

2 :  $\begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d)  $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

3 :  $\begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 5

Consider the function  $f(x) = \frac{1}{x^2 - kx}$ , where  $k$  is a nonzero constant. The derivative of  $f$  is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let  $k = 3$ , so that  $f(x) = \frac{1}{x^2 - 3x}$ . Write an equation for the line tangent to the graph of  $f$  at the point whose  $x$ -coordinate is 4.
- (b) Let  $k = 4$ , so that  $f(x) = \frac{1}{x^2 - 4x}$ . Determine whether  $f$  has a relative minimum, a relative maximum, or neither at  $x = 2$ . Justify your answer.
- (c) Find the value of  $k$  for which  $f$  has a critical point at  $x = -5$ .
- (d) Let  $k = 6$ , so that  $f(x) = \frac{1}{x^2 - 6x}$ . Find the partial fraction decomposition for the function  $f$ .

Find  $\int f(x) dx$ .

(a)  $f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4}$        $f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$

An equation for the line tangent to the graph of  $f$  at the point whose  $x$ -coordinate is 4 is  $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$ .

(b)  $f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2}$        $f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$

$f'(x)$  changes sign from positive to negative at  $x = 2$ .  
Therefore,  $f$  has a relative maximum at  $x = 2$ .

(c)  $f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$

(d)  $\frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left( \frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

2 :  $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

2 :  $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

1 : answer

4 :  $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 6

The Maclaurin series for a function  $f$  is given by  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$  and converges to  $f(x)$  for  $|x| < R$ , where  $R$  is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find  $R$ .
- (b) Write the first four nonzero terms of the Maclaurin series for  $f'$ , the derivative of  $f$ . Express  $f'$  as a rational function for  $|x| < R$ .
- (c) Write the first four nonzero terms of the Maclaurin series for  $e^x$ . Use the Maclaurin series for  $e^x$  to write the third-degree Taylor polynomial for  $g(x) = e^x f(x)$  about  $x = 0$ .

- (a) Let  $a_n$  be the  $n$ th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is  $R = \frac{1}{3}$ .

- (b) The first four nonzero terms of the Maclaurin series for  $f'$  are  $1 - 3x + 9x^2 - 27x^3$ .

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for  $e^x$  are  $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ .

The product of the Maclaurin series for  $e^x$  and the Maclaurin series for  $f'$  is

$$\left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left( x - \frac{3}{2}x^2 + 3x^3 - \dots \right)$$

$$= x - \frac{1}{2}x^2 + 2x^3 + \dots$$

The third-degree Taylor polynomial for  $g(x) = e^x f(x)$  about  $x = 0$  is  $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$ .

3 :  $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

3 :  $\begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$

3 :  $\begin{cases} 1 : \text{first four nonzero terms} \\ \quad \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$