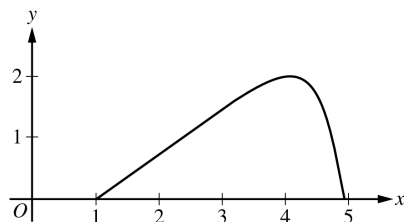


Part A (BC): Graphing calculator required**Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



For $0 \leq t \leq \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

Model Solution**Scoring**

- (a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

$$x''(1) = \left. \frac{d}{dt}(e^{\cos t}) \right|_{t=1} = -1.444407$$

$x''(1)$ with setup **1 point**

$$y(t) = 2\sin t \Rightarrow y'(t) = 2\cos t$$

$$y''(1) = \left. \frac{d}{dt}(2\cos t) \right|_{t=1} = -1.682942$$

$y''(1)$ with setup **1 point**

The acceleration vector at time $t = 1$ is

$$a(1) = \langle -1.444, -1.683 \text{ (or } -1.682) \rangle.$$

Scoring notes:

- The exact answer is $\langle x''(1), y''(1) \rangle = \langle -e^{\cos 1} \sin 1, -2\sin 1 \rangle$.
- $\langle -e^{\cos t} \sin t, -2\sin t \rangle$ together with an incorrect or missing evaluation at $t = 1$ earns 1 of the 2 points.

- A response of $\langle -e^{\cos t} \sin t, -2\sin t \rangle = \langle -e^{\cos 1} \sin 1, -2\sin 1 \rangle$ or equivalent earns only 1 of the 2 points because it equates an expression to a numerical value.
- An unsupported correct acceleration vector earns 1 of the 2 points.
- The acceleration vector may be presented with other symbols, for example $(\ , \)$ or $[\ , \]$.
- The components may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $x''(1) = -0.000828$ or -0.047433 and $y''(1) = -0.000609$ or -0.034905 . A response that presents one of these values with correct setups earns 1 of the 2 points.

Total for part (a) 2 points

- (b) For $0 \leq t \leq \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(e^{\cos t})^2 + (2\cos t)^2}$$

$$\sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5 \quad \text{1 point}$$

$$0 \leq t \leq \pi \text{ and } \sqrt{(e^{\cos t})^2 + (2\cos t)^2} = 1.5$$

$$\Rightarrow t = 1.254472, t = 2.358077$$

Answer **1 point**

The first time at which the speed of the particle is 1.5 is $t = 1.254$.

Scoring notes:

- A response with an implied equation is eligible for both points. For example, a response of “Speed = $\sqrt{(e^{\cos t})^2 + (2\cos t)^2}$ and is first equal to 1.5 at $t = 1.254$ ” earns both points.
- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5$ earns the first point. Speed = 1.5 by itself does not earn the first point. Both of these responses are eligible to earn the second point.
- A response need not consider the value $t = 2.358077$.
- A response of $t = 1.254$ alone does not earn either point.
- A response with a parenthesis error(s) in either $(e^{\cos t})^2$ or $(2\cos t)^2$ does not earn the first point but does earn the second point for the correct answer. Note: $\sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$ is not considered a parenthesis error.

- Degree mode: In degree mode, $\sqrt{(e^{\cos t})^2 + (2 \cos t)^2} = 1.5$ has no solution for $0 \leq t \leq \pi$.

A response that finds no time t at which the speed of the particle is 1.5 cannot be assumed to be working in degree mode.

Total for part (b) 2 points

- (c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x -coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{e^{\cos t}}$$

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{2 \cos 1}{e^{\cos 1}} = 0.629530$$

The slope of the line tangent to the curve at $t = 1$ is 0.630 (or 0.629).

$$x(1) = x(0) + \int_0^1 \frac{dx}{dt} dt = 1 + \int_0^1 e^{\cos t} dt = 3.341575$$

The x -coordinate of the position at $t = 1$ is 3.342 (or 3.341).

Slope with supporting work **1 point**

$$\int_0^1 e^{\cos t} dt$$

$x(1)$ **1 point**

Scoring notes:

- To earn the first point, the response must communicate $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; for example:
 - $\frac{dy}{dx} = \frac{2 \cos 1}{e^{\cos 1}}$
 - $\frac{dy/dt}{dx/dt} = 0.63$
 - $x'(1) = 1.716526$, $y'(1) = 1.080605$, slope = 0.63
 - $\frac{dy}{dt} = 2 \cos t$, slope = 0.63
- A response may import an incorrect expression for $y'(t)$ or value of $y'(1)$ from part (a), provided it was declared in part (a).
- The second point is earned for a response that presents the definite integral $\int_0^1 e^{\cos t} dt$ or $\int_0^1 \frac{dx}{dt} dt$ with or without the initial condition.

- For the second point, if the differential is missing:

- $\int_0^1 e^{\cos t} dt$ earns the second point and is eligible for the third point.

- $x(1) = \int_0^1 e^{\cos t} dt$ earns the second point but is not eligible for the third point.

- $x(1) = 1 + \int_0^1 e^{\cos t} dt$ earns the second point and is eligible for the third point.

- $x(1) = \int_0^1 e^{\cos t} + 1$ does not earn the second point but earns the third point for the correct answer.

- The third point is not earned for a response that presents an incorrect statement, such as

$$x(1) = \int_0^1 e^{\cos t} dt = 1 + 2.342.$$

- Degree mode: In degree mode, $\frac{dy}{dx} = 0.735759$ or 0.012841 and $1 + \int_0^1 e^{\cos t} dt = 3.718144$.

Total for part (c) 3 points

- (d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq \pi$. Show the setup for your calculations.

$$\int_0^\pi \sqrt{(e^{\cos t})^2 + (2 \cos t)^2} dt$$

$$= 6.034611$$

The total distance traveled by the particle over $0 \leq t \leq \pi$ is 6.035 (or 6.034).

Integral **1 point**

Answer **1 point**

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- Parentheses errors were assessed in part (b) and, therefore, will not affect the scoring in part (d).
- If the integrand is an incorrect speed function imported from part (b), the response earns the first point and does not earn the second point.
- An unsupported answer of 6.035 (or 6.034) does not earn either point.
- Degree mode: In degree mode, the total distance is 10.596835 or 8.536161.

Total for part (d) 2 points

Total for question 2 9 points

Part A (BC): Graphing calculator required**Question 2****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy -plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1+t^2}$ and $\frac{dy}{dt} = \ln(2+t^2)$. At time $t = 4$, the particle is at the point $(1, 5)$.

Model Solution**Scoring**

- (a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

$$\left. \frac{dy}{dx} \right|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$$

Answer

1 point

The slope of the line tangent to the path of the particle at time $t = 4$ is 0.701.

Scoring notes:

- To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point: $\frac{y'(4)}{x'(4)} = 0.701$, $\frac{\ln(2+4^2)}{\sqrt{1+4^2}}$, or $\frac{\ln 18}{\sqrt{17}}$.
- Note: A response with an incorrect equation of the form “function = constant”, such as $\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}$, will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.

Total for part (a) 1 point

- (b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

$$\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$$

Speed

1 point

The speed of the particle at time $t = 4$ is 5.035.

$$a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$$

First component of acceleration

1 point

The acceleration vector of the particle at time $t = 4$ is $\langle 0.970, 0.444 \rangle$.

Second component of acceleration

1 point**Scoring notes:**

- To earn any of these points, the setup used to perform the calculation must be evident in the response. For example, $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$ or $\sqrt{17 + (\ln 18)^2}$ earns the first point, and $\langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$ earns both the second and third points.
- The second and third points can be earned independently.
- If the acceleration vector is not presented as an ordered pair, the x - and y -components must be labeled.
- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.
- A response which correctly calculates expressions for both $x''(t) = \frac{t}{\sqrt{1+t^2}}$ and $y''(t) = \frac{2t}{2+t^2}$, but which fails to evaluate both of these expressions at $t = 4$, earns only 1 of the last 2 points.
- An unsupported acceleration vector earns only 1 of the last 2 points.

Total for part (b) 3 points

(c) Find the y -coordinate of the particle's position at time $t = 6$.

$y(6) = y(4) + \int_4^6 \ln(2 + t^2) dt$	Integrand	1 point
	Uses $y(4)$	1 point
$= 5 + 6.570517 = 11.570517$	Answer	1 point
The y -coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570).		

Scoring notes:

- For the first point, an integrand of $\ln(2 + t^2)$ can appear in either an indefinite integral or an incorrect definite integral.
- A definite integral with incorrect limits is not eligible for the answer point.
- Similarly, an indefinite integral is not eligible for the answer point.
- For the second point, the value for $y(4)$ must be added to a definite integral.
- A response that reports the correct x -coordinate of the particle's position at time $t = 6$ as $x(6) = x(4) + \int_4^6 \sqrt{1 + t^2} dt = 11.200$ (or 11.201) instead of the y -coordinate, earns 2 out of the 3 points.
- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).
- If the differential is missing:
 - $y(6) = \int_4^6 \ln(2 + t^2)$ earns the first point and is eligible for the third.
 - $y(6) = \int_4^6 \ln(2 + t^2) + y(4)$ does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
 - $y(6) = y(4) + \int_4^6 \ln(2 + t^2)$ earns the first two points and is eligible for the third.

Total for part (c) 3 points(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

$\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$	Integrand	1 point
$= 12.136228$	Answer	1 point
The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136.		

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- To earn the second point, a response must have earned the first point and must present the value 12.136.
- An unsupported answer of 12.136 does not earn either point.

Total for part (d) 2 points**Total for question 2 9 points**

AP[®] Calculus BC

2016 Scoring Guidelines

© 2016 The College Board. College Board, Advanced Placement Program, AP, AP Central, and the acorn logo are registered trademarks of the College Board.

Visit the College Board on the Web: www.collegeboard.org.

AP Central is the official online home for the AP Program: apcentral.collegeboard.org.

2016 SCORING GUIDELINES

Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \leq t \leq 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

- Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.
- Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- For $0 \leq t \leq 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

$$(a) \quad R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120 \text{ liters/hr}^2$$

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$$

$$(b) \quad \text{The total amount of water removed is given by } \int_0^8 R(t) \, dt.$$

$$\begin{aligned} \int_0^8 R(t) \, dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

$$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$$

$$(c) \quad \begin{aligned} \text{Total} &\approx 50000 + \int_0^8 W(t) \, dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters} \end{aligned}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$$

$$(d) \quad W(0) - R(0) > 0, \quad W(8) - R(8) < 0, \quad \text{and } W(t) - R(t) \text{ is continuous.}$$

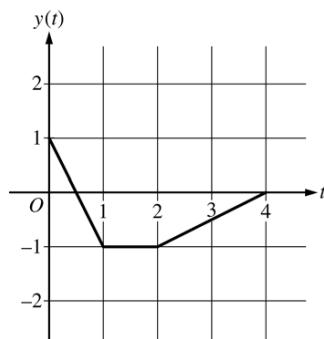
$$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$$

Therefore, the Intermediate Value Theorem guarantees at least one time t , $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t , the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

2016 SCORING GUIDELINES

Question 2



At time t , the position of a particle moving in the xy -plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y , consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

- Find the position of the particle at $t = 3$.
- Find the slope of the line tangent to the path of the particle at $t = 3$.
- Find the speed of the particle at $t = 3$.
- Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $x(3) = x(0) + \int_0^3 x'(t) dt = 5 + 9.377035 = 14.377$

$$y(3) = -\frac{1}{2}$$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

(b) Slope = $\frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$

(c) Speed = $\sqrt{(x'(3))^2 + (y'(3))^2} = 9.969$ (or 9.968)

(d) Distance = $\int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt$
 $= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2} dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2} dt$
 $= 2.237871 + 2.112003 = 4.350$ (or 4.349)

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

1 : slope

2 : $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

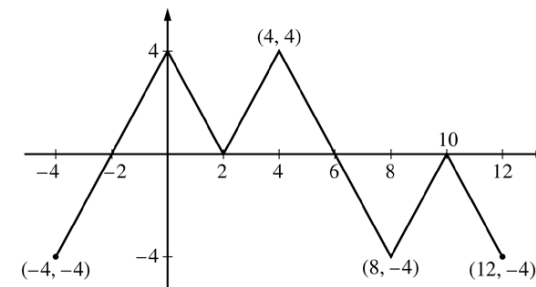
2016 SCORING GUIDELINES

Question 3

The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by

$$g(x) = \int_2^x f(t) dt.$$

- Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.



Graph of f

- The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \leq 0$ for $8 \leq x \leq 12$.
- The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 6$.
- $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

- (d) $g(x) \leq 0$ for $-4 \leq x \leq 2$ and $10 \leq x \leq 12$.

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 : $\begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$

2 : intervals

2016 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

- (a) Find $\frac{d^2y}{dx^2}$ in terms of x and y .
- (b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.
- (c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.
- (d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$

2 : $\frac{d^2y}{dx^2}$ in terms of x and y

(b) $\left. \frac{dy}{dx} \right|_{(x,y)=(-2,8)} = (-2)^2 - \frac{1}{2} \cdot 8 = 0$

2 : conclusion with justification

$\left. \frac{d^2y}{dx^2} \right|_{(x,y)=(-2,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2} \cdot 8\right) = -4 < 0$

Thus, the graph of f has a relative maximum at the point $(-2, 8)$.

(c) $\lim_{x \rightarrow -1} (g(x) - 2) = 0$ and $\lim_{x \rightarrow -1} 3(x+1)^2 = 0$

3 : $\begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Using L'Hospital's Rule,

$\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right) = \lim_{x \rightarrow -1} \left(\frac{g'(x)}{6(x+1)} \right)$

$\lim_{x \rightarrow -1} g'(x) = 0$ and $\lim_{x \rightarrow -1} 6(x+1) = 0$

Using L'Hospital's Rule,

$\lim_{x \rightarrow -1} \left(\frac{g'(x)}{6(x+1)} \right) = \lim_{x \rightarrow -1} \left(\frac{g''(x)}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$

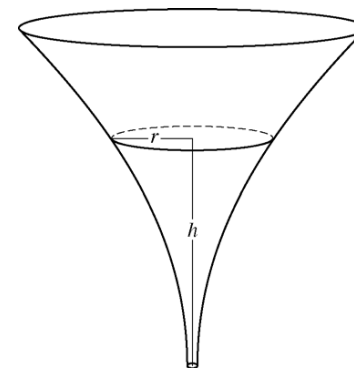
(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1) \cdot \frac{1}{2} = \frac{3}{2}$

$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{5}{4}$

2 : $\begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

2016 SCORING GUIDELINES

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h , the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \leq h \leq 10$. The units of r and h are inches.

- (a) Find the average value of the radius of the funnel.
- (b) Find the volume of the funnel.
- (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(a) Average radius $= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[3h + \frac{h^3}{3} \right]_0^{10}$
 $= \frac{1}{200} \left(\left(30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60}$ in

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^{10} \left(\left(\frac{1}{20} \right) (3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh$
 $= \frac{\pi}{400} \left[9h + 2h^3 + \frac{h^5}{5} \right]_0^{10}$
 $= \frac{\pi}{400} \left(\left(90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40}$ in³

3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dr}{dt} = \frac{1}{20}(2h) \frac{dh}{dt}$
 $-\frac{1}{5} = \frac{3}{10} \frac{dh}{dt}$
 $\frac{dh}{dt} = -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3}$ in/sec

3 : $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2016 SCORING GUIDELINES

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence.

It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the n th derivative of f at $x = 1$ is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
- (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.
- (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1$, $f'(1) = -\frac{1}{2}$, $f''(1) = \frac{1}{2^2}$, $f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots$$

$$+ \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$$

- (b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.

Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

$$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

- (d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$$

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$