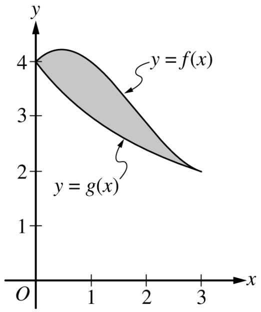
Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

Model Solution	Scoring	

(a) Find the area of the shaded region enclosed by the graphs of f and g.

Area $= \int_0^3 (f(x) - g(x))\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx$	Integrand	**1 point**
$= 10 - \int_0^3 \dfrac{12}{3+x}\,dx = 10 - 12\left[\ln\lvert 3+x\rvert\right]_0^3$	Antiderivative of $g(x)$	**1 point**
$= 10 - 12(\ln 6 - \ln 3) = 10 - 12(\ln 2)$	Answer	**1 point**

Scoring notes:

- The first point is earned for any of the integrands $f(x) - g(x)$, $g(x) - f(x)$, $\lvert f(x) - g(x)\rvert$, or $\lvert g(x) - f(x)\rvert$ in any definite integral. If the limits are incorrect, the response does not earn the third point.
- The first point is earned with an implied integrand for f and explicit integrand for g, such as $10 - \int_0^3 g(x)\,dx$.
- The second point is earned for finding $a\int \dfrac{dx}{3+x} = a \cdot \ln\lvert 3+x\rvert$ or $a \cdot \ln(3+x)$.
- A response is eligible for the third point only if it has earned the first 2 points. The third point is earned only for the correct answer. The answer does not need to be simplified; however, if simplification is attempted, it must be correct.
- A response is not eligible for the third point with incorrect limits of integration for u-substitution, for example, $\int_0^3 \dfrac{12}{3+x}\,dx = \int_0^3 \dfrac{12}{u}\,du = 12\left[\ln(x+3)\right]_0^3$.
- A response with incorrect communication, such as "Area $= \int_0^3 (g(x) - f(x))\,dx = 10 - 12(\ln 2)$," does not earn the third point. However, a response of "$\int_0^3 (g(x) - f(x))\,dx = 12(\ln 2) - 10$, so the area is $10 - 12(\ln 2)$" earns all 3 points.

Total for part (a) 3 points

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

$\int_0^\infty (g(x))^2\,dx = \lim\limits_{b\to\infty} \int_0^b \dfrac{144}{(3+x)^2}\,dx$	Limit notation	**1 point**
$= \lim\limits_{b\to\infty}\left(-\dfrac{144}{(3+x)}\Bigg\rvert_0^b\right)$	Antiderivative	**1 point**
$= \lim\limits_{b\to\infty}\left(-\dfrac{144}{3+b} + \dfrac{144}{3}\right) = 48$	Answer	**1 point**

Scoring notes:

- To earn the first point a response must correctly use limit notation throughout the problem and not include arithmetic with infinity, for example, $\left[-\dfrac{144}{3+x}\right]_0^\infty$ or $-\dfrac{144}{3+\infty} + 48$.
- The second point can be earned by finding an antiderivative of the form $-\dfrac{a}{(3+x)}$ for $a > 0$, from an indefinite or improper integral, with or without correct limit notation. If $a \ne 144$, the response does not earn the third point.
- The third point is earned only for an answer of 48 (or equivalent).
- A response is not eligible for the third point with incorrect limits of integration for u-substitution, for example, $\lim\limits_{b\to\infty}\int_0^b \dfrac{144}{u^2}\,du = \lim\limits_{b\to\infty}\left[-\dfrac{144}{3+x}\right]_0^b$.

Total for part (b) 3 points

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Using integration by parts, $u = x \quad dv = f'(x)\,dx$ $du = dx \quad v = f(x)$	u and dv	**1 point**
$\int h(x)\,dx = \int x \cdot f'(x)\,dx = x \cdot f(x) - \int f(x)\,dx$	$\int h(x)\,dx$ $= x \cdot f(x) - \int f(x)\,dx$	**1 point**
$\int_0^3 h(x)\,dx = \int_0^3 x \cdot f'(x)\,dx = x \cdot f(x)\Big\vert_0^3 - \int_0^3 f(x)\,dx$ $= (3 \cdot f(3) - 0 \cdot f(0)) - 10 = 3 \cdot 2 - 0 - 10 = -4$	Answer	**1 point**

Scoring notes:

- The first and second points are earned with an implied u and dv in the presence of $x \cdot f(x) - \int f(x)\,dx$ or $x \cdot f(x)\Big\vert_0^3 - 10.$
- Limits of integration may be present, omitted, or partially present in the work for the first and second points.
- The tabular method may be used to show integration by parts. In this case, the first point is earned by having columns (labeled or unlabeled) that begin with x and $f'(x)$. The second point is earned for $x \cdot f(x) - \int f(x)\,dx.$
- The third point is earned only for the correct answer and can only be earned if the first 2 points were earned.

Total for part (c) 3 points

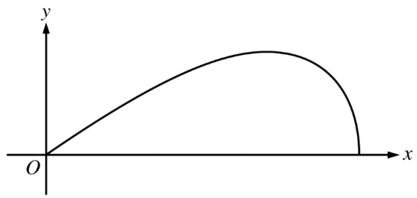
Total for question 5 9 points

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

Model Solution	Scoring	

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

$6x\sqrt{4-x^2} = 0 \Rightarrow x = 0,\ x = 2$ Area $= \int_0^2 6x\sqrt{4-x^2}\,dx$	Integrand	**1 point**
Let $u = 4 - x^2$. $du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4-x^2}\,dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\,du = -3\int_4^0 u^{1/2}\,du = 3\int_0^4 u^{1/2}\,du$ $= 2u^{3/2}\Big\vert_{u=0}^{u=4} = 2\cdot 8 = 16$	Antiderivative	**1 point**
The area of the region is 16 square inches.	Answer	**1 point**

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4-x^2}$ or $6x\sqrt{4-x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4-x^2}$, for any nonzero constant A. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u-substitution and have incorrect limits of integration or do not change the limits of integration from x- to u-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

(b) It is known that, for $y = cx\sqrt{4-x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4-x^2}$ has its maximum on the interval $0 < x < 2$. $\frac{dy}{dx} = \frac{c\left(4-2x^2\right)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\frac{dy}{dx} = 0$	**1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4-\left(\sqrt{2}\right)^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer	**1 point**

Scoring notes:

- The first point is earned for setting $\frac{dy}{dx} = 0$, $\frac{c\left(4-2x^2\right)}{\sqrt{4-x^2}} = 0$, or $c\left(4-2x^2\right) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$\text{Volume} = \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2\left(4-x^2\right) dx$	Form of the integrand	**1 point**
	Limits and constant	**1 point**
$= \pi c^2 \int_0^2 \left(4x^2 - x^4\right) dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \Big\vert_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2 \left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting an integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A. Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π, the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2 \left(cx\sqrt{4-x^2}\right)^2 dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ for any nonzero A. If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points