

**Part B (AB or BC): Graphing calculator not allowed****Question 3****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A student starts reading a book at time  $t = 0$  minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function  $R$ , where  $R(t)$  is measured in words per minute. Selected values of  $R(t)$  are given in the table shown.

|                           |    |     |     |     |
|---------------------------|----|-----|-----|-----|
| $t$ (minutes)             | 0  | 2   | 8   | 10  |
| $R(t)$ (words per minute) | 90 | 100 | 150 | 162 |

**Model Solution****Scoring**

- A** Approximate  $R'(1)$  using the average rate of change of  $R$  over the interval  $0 \leq t \leq 2$ . Show the work that leads to your answer. Indicate units of measure.

$$R'(1) \approx \frac{R(2) - R(0)}{2 - 0}$$

$$= \frac{100 - 90}{2} = \frac{10}{2} = 5 \text{ words per minute per minute}$$

Answer with setup **Point 1 (P1)**Units **Point 2 (P2)****Scoring Notes for Part A**

- To earn **P1**, a response must present the answer along with the supporting work of a difference and a quotient using values from the table.
  - $\frac{100 - 90}{2 - 0}$ ,  $\frac{10}{2 - 0}$ ,  $\frac{100 - 90}{2}$ , or  $\frac{R(2) - R(0)}{2 - 0} = 5$  is sufficient to earn **P1**.
  - $\frac{R(2) - R(0)}{2 - 0}$  by itself is not sufficient to earn **P1**.
- P2** is earned for correct units, whether or not they are attached to a numerical value for the average rate of change.
- P2** is also earned for the units “words/minute<sup>2</sup>.”

- B** Must there be a value  $c$ , for  $0 < c < 10$ , such that  $R(c) = 155$ ? Justify your answer.

 $R$  is differentiable implies  $R$  is continuous.Differentiable  
implies continuous**Point 3 (P3)**

$$R(0) = 90 < 155 < R(10) = 162$$

Answer with  
justification**Point 4 (P4)**

Therefore, by the Intermediate Value Theorem, there must be a value  $c$ , with  $0 < c < 10$ , such that  $R(c) = 155$ .

**Scoring Notes for Part B**

- To earn **P3**, a response must state that  $R$  is continuous because  $R$  is differentiable (or equivalent). A response that simply states “ $R$  is continuous” without justification does not earn **P3**.
- A response does not need to earn **P3** to be eligible for **P4**.
- To earn **P4**, a response must indicate that  $R(0) < 155$  (or  $R(2) < 155$  or  $R(8) < 155$ ) and  $R(10) > 155$ , state that “ $R$  is continuous,” and answer “yes” in some way.
- To earn **P4**, a response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

- C** Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of  $\int_0^{10} R(t) dt$ . Show the work that leads to your answer.

|                                                                                                                                                                              |                             |                     |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------|---------------------|
| $\int_0^{10} R(t) dt$ $\approx \frac{R(0) + R(2)}{2}(2 - 0) + \frac{R(2) + R(8)}{2}(8 - 2)$ $+ \frac{R(8) + R(10)}{2}(10 - 8)$                                               | Form of trapezoidal sum     | <b>Point 5 (P5)</b> |
| $= \frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ $= \frac{190}{2}(2) + \frac{250}{2}(6) + \frac{312}{2}(2) = 190 + 750 + 312 = 1252$ | Answer with supporting work | <b>Point 6 (P6)</b> |

#### Scoring Notes for Part C

- Read “=” as “ $\approx$ ” for **P5**.
- The form of a trapezoidal sum includes three terms, each of which includes a product of two factors, where one of the factors incorporates the  $\frac{1}{2}$  as part of the product. To earn **P5**, at least five of the six factors must be correct. If any of the six factors is incorrect, the response does not earn **P6**. Consider the following examples:
  - $\frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$  earns **P5** and is sufficient to earn **P6**.
  - $\frac{190}{2}(2) + \frac{250}{2}(6) + \frac{312}{2}(2)$  earns **P5** and is sufficient to earn **P6**.
  - $\frac{1}{2}((R(0) + R(2))(2) + (R(2) + R(8))(6) + (R(8) + R(10))(2))$  earns **P5** and is eligible for **P6**.
  - $\frac{90 + 100}{2}(2) + \frac{100 + 150}{2}(2) + \frac{150 + 162}{2}(2)$  earns **P5** but is not eligible for **P6**.  
(Note that the factor of 2 in the second term of this expression is incorrect.)
- Special case:** A response of  $(90 + 100) + (100 + 150)3 + (150 + 162)$  earns both **P5** and **P6**.
- To be eligible for **P6**, a response must have earned **P5**.  
**Special case:** A response of  $95 \cdot 2 + 125 \cdot 6 + 156 \cdot 2$  earns **P6** but does not earn **P5**.
- A response of  $\frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$  or equivalent banks **P6** (i.e., subsequent errors in simplification will not be considered in scoring for **P6**).
- A response of  $\frac{(90 \cdot 2 + 100 \cdot 6 + 150 \cdot 2) + (100 \cdot 2 + 150 \cdot 6 + 162 \cdot 2)}{2}$  or equivalent earns both **P5** and **P6**. (Note that the average of the left Riemann sum and right Riemann sum is equivalent to the trapezoidal sum.)
- A completely correct left Riemann sum (e.g.,  $90 \cdot 2 + 100 \cdot 6 + 150 \cdot 2 = 1080$ ) or a completely correct right Riemann sum (e.g.,  $100 \cdot 2 + 150 \cdot 6 + 162 \cdot 2 = 1424$ ) earns **P5** but does not earn **P6**.

- D** A teacher also starts reading at time  $t = 0$  minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function  $W$  defined by  $W(t) = -\frac{3}{10}t^2 + 8t + 100$ , where  $W(t)$  is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

|                                                                                                                                                   |                |                     |
|---------------------------------------------------------------------------------------------------------------------------------------------------|----------------|---------------------|
| $\int_0^{10} W(t) dt = \int_0^{10} \left(-\frac{3}{10}t^2 + 8t + 100\right) dt$                                                                   | Integrand      | <b>Point 7 (P7)</b> |
| $= \left(-\frac{1}{10}t^3 + 4t^2 + 100t\right)\bigg _0^{10}$                                                                                      | Antiderivative | <b>Point 8 (P8)</b> |
| $= \left(-\frac{1}{10} \cdot 1000 + 4 \cdot 100 + 100 \cdot 10\right) - \left(-\frac{1}{10} \cdot 0 + 4 \cdot 0 + 100 \cdot 0\right)$<br>$= 1300$ | Answer         | <b>Point 9 (P9)</b> |

Based on the model, the teacher has read 1300 words by the end of the 10 minutes.

#### Scoring Notes for Part D

- P7** is earned for an indefinite or definite integral with integrand  $W(t)$ , with or without the differential  $dt$ .
- P8** is earned for the correct antiderivative, with or without the constant of integration.
- To be eligible for **P9**, a response must have earned **P8**.
- A response of  $\left(-\frac{1}{10} \cdot 1000 + 4 \cdot 100 + 100 \cdot 10\right) - \left(-\frac{1}{10} \cdot 0 + 4 \cdot 0 + 100 \cdot 0\right)$  or equivalent banks **P9** (i.e., subsequent errors in simplification will not be considered in scoring for **P9**).

2019

AP<sup>®</sup> CollegeBoard

# AP<sup>®</sup> Calculus BC

## Scoring Guidelines

## 2019 SCORING GUIDELINES

## Question 1

(a)  $\int_0^5 E(t) dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

(b)  $\frac{1}{5-0} \int_0^5 L(t) dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

- (c) The rate of change in the number of fish in the lake at time  $t$  is given by  $E(t) - L(t)$ .

$$E(t) - L(t) = 0 \Rightarrow t = 6.20356$$

$E(t) - L(t) > 0$  for  $0 \leq t < 6.20356$ , and  $E(t) - L(t) < 0$  for  $6.20356 < t \leq 8$ . Therefore the greatest number of fish in the lake is at time  $t = 6.204$  (or 6.203).

— OR —

Let  $A(t)$  be the change in the number of fish in the lake from midnight to  $t$  hours after midnight.

$$A(t) = \int_0^t (E(s) - L(s)) ds$$

$$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$$

| $t$ | $A(t)$    |
|-----|-----------|
| 0   | 0         |
| $C$ | 135.01492 |
| 8   | 80.91998  |

Therefore the greatest number of fish in the lake is at time  $t = 6.204$  (or 6.203).

(d)  $E'(5) - L'(5) = -10.7228 < 0$

Because  $E'(5) - L'(5) < 0$ , the rate of change in the number of fish is decreasing at time  $t = 5$ .

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 :  $\begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

2 :  $\begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

## 2019 SCORING GUIDELINES

## Question 2

(a)  $\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = 3.534292$

The area of  $S$  is 3.534.

(b)  $\frac{1}{\sqrt{\pi}-0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.579933$

The average distance from the origin to a point on the curve  $r = r(\theta)$  for  $0 \leq \theta \leq \sqrt{\pi}$  is 1.580 (or 1.579).

(c)  $\tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$

$$\frac{1}{2} \int_0^{\tan^{-1} m} (r(\theta))^2 d\theta = \frac{1}{2} \left( \frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta \right)$$

- (d) As  $k \rightarrow \infty$ , the circle  $r = k \cos \theta$  grows to enclose all points to the right of the  $y$ -axis.

$$\begin{aligned} \lim_{k \rightarrow \infty} A(k) &= \frac{1}{2} \int_0^{\pi/2} (r(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} (3\sqrt{\theta} \sin(\theta^2))^2 d\theta = 3.324 \end{aligned}$$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 :  $\begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \text{applied to } m \text{ as limit of} \\ \text{integration} \\ 1 : \text{equation} \end{cases}$

2 :  $\begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

## 2019 SCORING GUIDELINES

## Question 3

$$\begin{aligned} \text{(a)} \quad \int_{-6}^5 f(x) \, dx &= \int_{-6}^{-2} f(x) \, dx + \int_{-2}^5 f(x) \, dx \\ &\Rightarrow 7 = \int_{-6}^{-2} f(x) \, dx + 2 + \left(9 - \frac{9\pi}{4}\right) \\ &\Rightarrow \int_{-6}^{-2} f(x) \, dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_3^5 (2f'(x) + 4) \, dx &= 2 \int_3^5 f'(x) \, dx + \int_3^5 4 \, dx \\ &= 2(f(5) - f(3)) + 4(5 - 3) \\ &= 2(0 - (3 - \sqrt{5})) + 8 \\ &= 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5} \end{aligned}$$

— OR —

$$\begin{aligned} \int_3^5 (2f'(x) + 4) \, dx &= [2f(x) + 4x]_{x=3}^{x=5} \\ &= (2f(5) + 20) - (2f(3) + 12) \\ &= (2 \cdot 0 + 20) - (2(3 - \sqrt{5}) + 12) \\ &= 2 + 2\sqrt{5} \end{aligned}$$

$$\text{(c)} \quad g'(x) = f(x) = 0 \Rightarrow x = -1, x = \frac{1}{2}, x = 5$$

| $x$           | $g(x)$                |
|---------------|-----------------------|
| -2            | 0                     |
| -1            | $\frac{1}{2}$         |
| $\frac{1}{2}$ | $-\frac{1}{4}$        |
| 5             | $11 - \frac{9\pi}{4}$ |

On the interval  $-2 \leq x \leq 5$ , the absolute maximum value of  $g$  is  $g(5) = 11 - \frac{9\pi}{4}$ .

$$\begin{aligned} \text{(d)} \quad \lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} &= \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \\ &= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}} \end{aligned}$$

$$3 : \begin{cases} 1 : \int_{-6}^5 f(x) \, dx = \int_{-6}^{-2} f(x) \, dx + \int_{-2}^5 f(x) \, dx \\ 1 : \int_{-2}^5 f(x) \, dx \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$$

1 : answer

## 2019 SCORING GUIDELINES

## Question 4

$$\begin{aligned} \text{(a)} \quad V &= \pi r^2 h = \pi(1)^2 h = \pi h \\ \frac{dV}{dt} \Big|_{h=4} &= \pi \frac{dh}{dt} \Big|_{h=4} = \pi \left( -\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d^2 h}{dt^2} &= -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left( -\frac{1}{10} \sqrt{h} \right) = \frac{1}{200} \\ \text{Because } \frac{d^2 h}{dt^2} &= \frac{1}{200} > 0 \text{ for } h > 0, \text{ the rate of change of the} \\ \text{height is increasing when the height of the water is } 3 \text{ feet.} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{dh}{\sqrt{h}} &= -\frac{1}{10} dt \\ \int \frac{dh}{\sqrt{h}} &= \int -\frac{1}{10} dt \\ 2\sqrt{h} &= -\frac{1}{10} t + C \\ 2\sqrt{5} &= -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5} \\ 2\sqrt{h} &= -\frac{1}{10} t + 2\sqrt{5} \\ h(t) &= \left( -\frac{1}{20} t + \sqrt{5} \right)^2 \end{aligned}$$

$$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{d}{dh} \left( -\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration

## 2019 SCORING GUIDELINES

## Question 5

$$(a) f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2}$$

$$f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

$$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$$

$$(b) \frac{1}{x^2-2x-8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$\Rightarrow 1 = A(x+2) + B(x-4)$$

$$\Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}$$

$$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$$

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left( \frac{1}{x-4} - \frac{1}{x+2} \right) dx \\ &= \left[ \frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1} \\ &= \left( \frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left( \frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \end{aligned}$$

$$(c) \int_0^2 \frac{1}{x^2-2x+1} dx = \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx$$

$$= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx$$

$$= \lim_{b \rightarrow 1^-} \left( -\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \rightarrow 1^+} \left( -\frac{1}{x-1} \Big|_{x=b}^{x=2} \right)$$

$$= \lim_{b \rightarrow 1^-} \left( -\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left( -1 + \frac{1}{b-1} \right)$$

Because  $\lim_{b \rightarrow 1^-} \left( -\frac{1}{b-1} \right)$  does not exist, the integral diverges.

$$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$$

## 2019 SCORING GUIDELINES

## Question 6

$$(a) f(0) = 3 \text{ and } f'(0) = -2$$

The third-degree Taylor polynomial for  $f$  about  $x = 0$  is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-2}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$$

$$(b) \text{ The first three nonzero terms of the Maclaurin series for } e^x \text{ are}$$

$$1 + x + \frac{1}{2!}x^2.$$

$$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$$

The second-degree Taylor polynomial for  $e^x f(x)$  about  $x = 0$  is

$$\begin{aligned} 3 \left( 1 + x + \frac{1}{2!}x^2 \right) - 2x(1+x) + \frac{3}{2}x^2(1) \\ = 3 + (3-2)x + \left( \frac{3}{2} - 2 + \frac{3}{2} \right)x^2 \\ = 3 + x + x^2. \end{aligned}$$

$$(c) h(1) = \int_0^1 f(t) dt$$

$$\approx \int_0^1 \left( 3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3 \right) dt$$

$$= \left[ 3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4 \right]_{t=0}^{t=1}$$

$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

$$(d) \text{ The alternating series error bound is the absolute value of the first omitted term of the series for } h(1).$$

$$\int_0^1 \left( \frac{54}{4!}t^4 \right) dt = \left[ \frac{9}{20}t^5 \right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \leq \left| \frac{9}{20} \right| = 0.45$$

$$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$$

# AP Calculus BC

## Scoring Guidelines

### 2018 SCORING GUIDELINES

#### Question 1

(a)  $\int_0^{300} r(t) \, dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval  $0 \leq t \leq 300$ .

(b)  $20 + \int_0^{300} (r(t) - 0.7) \, dt = 20 + \int_0^{300} r(t) \, dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time  $t = 300$ .

(c) Based on part (b), the number of people in line at time  $t = 300$  is 80.

The first time  $t$  that there are no people in line is

$$300 + \frac{80}{0.7} = 414.286 \text{ (or 414.285) seconds.}$$

(d) The total number of people in line at time  $t$ ,  $0 \leq t \leq 300$ , is modeled by  $20 + \int_0^t r(x) \, dx - 0.7t$ .

$$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$$

| $t$   | People in line for escalator |
|-------|------------------------------|
| 0     | 20                           |
| $t_1$ | 3.803                        |
| $t_2$ | 158.070                      |
| 300   | 80                           |

The number of people in line is a minimum at time  $t = 33.013$  seconds, when there are 4 people in line.

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 :  $\begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

1 : answer

4 :  $\begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

## 2018 SCORING GUIDELINES

## Question 2

(a)  $p'(25) = -1.179$

At a depth of 25 meters, the density of plankton cells is changing at a rate of  $-1.179$  million cells per cubic meter per meter.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{meaning with units} \end{cases}$$

(b)  $\int_0^{30} 3p(h) \, dh = 1675.414936$

There are 1675 million plankton cells in the column of water between  $h = 0$  and  $h = 30$  meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(c)  $\int_{30}^K 3f(h) \, dh$  represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of  $K$  meters.

The number of plankton cells, in millions, in the entire column of water is given by  $\int_0^{30} 3p(h) \, dh + \int_{30}^K 3f(h) \, dh$ .

Because  $0 \leq f(h) \leq u(h)$  for all  $h \geq 30$ ,

$$3 \int_{30}^K f(h) \, dh \leq 3 \int_{30}^K u(h) \, dh \leq 3 \int_{30}^{\infty} u(h) \, dh = 3 \cdot 105 = 315.$$

The total number of plankton cells in the column of water is bounded by  $1675.415 + 315 = 1990.415 \leq 2000$  million.

$$3 : \begin{cases} 1 : \text{integral expression} \\ 1 : \text{compares improper integral} \\ 1 : \text{explanation} \end{cases}$$

(d)  $\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = 757.455862$

The total distance traveled by the boat over the time interval  $0 \leq t \leq 1$  is 757.456 (or 757.455) meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{total distance} \end{cases}$$

## 2018 SCORING GUIDELINES

## Question 3

(a)  $f(-5) = f(1) + \int_1^{-5} g(x) \, dx = f(1) - \int_{-5}^1 g(x) \, dx$   
 $= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b)  $\int_1^6 g(x) \, dx = \int_1^3 g(x) \, dx + \int_3^6 g(x) \, dx$   
 $= \int_1^3 2 \, dx + \int_3^6 2(x-4)^2 \, dx$   
 $= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$

$$3 : \begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$$

(c) The graph of  $f$  is increasing and concave up on  $0 < x < 1$  and  $4 < x < 6$  because  $f'(x) = g(x) > 0$  and  $f''(x) = g(x)$  is increasing on those intervals.

$$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$$

(d) The graph of  $f$  has a point of inflection at  $x = 4$  because  $f''(x) = g(x)$  changes from decreasing to increasing at  $x = 4$ .

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$$

## 2018 SCORING GUIDELINES

## Question 4

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$  is the rate at which the height of the tree is changing, in meters per year, at time  $t = 6$  years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because  $H$  is differentiable on  $3 \leq t \leq 5$ ,  $H$  is continuous on  $3 \leq t \leq 5$ .

By the Mean Value Theorem, there exists a value  $c$ ,  $3 < c < 5$ , such that  $H'(c) = 2$ .

$$(c) \quad \text{The average height of the tree over the time interval } 2 \leq t \leq 10 \text{ is given by } \frac{1}{10 - 2} \int_2^{10} H(t) \, dt.$$

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) \, dt &\approx \frac{1}{8} \left( \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval  $2 \leq t \leq 10$  is  $\frac{263}{32}$  meters.

$$(d) \quad G(x) = 50 \Rightarrow x = 1$$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is  $\frac{3}{4}$  meter per year.

$$2: \begin{cases} 1: \text{estimate} \\ 1: \text{interpretation with units} \end{cases}$$

$$2: \begin{cases} 1: \frac{H(5) - H(3)}{5 - 3} \\ 1: \text{conclusion using Mean Value Theorem} \end{cases}$$

$$2: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{approximation} \end{cases}$$

$$3: \begin{cases} 2: \frac{d}{dt}(G(x)) \\ 1: \text{answer} \end{cases}$$

Note: max 1/3 [1-0] if no chain rule

## 2018 SCORING GUIDELINES

## Question 5

$$(a) \quad \text{Area} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} (4^2 - (3 + 2\cos \theta)^2) \, d\theta$$

$$(b) \quad \frac{dr}{d\theta} = -2\sin \theta \Rightarrow \left. \frac{dr}{d\theta} \right|_{\theta=\pi/2} = -2$$

$$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$$

$$y = r\sin \theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin \theta + r\cos \theta$$

$$x = r\cos \theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos \theta - r\sin \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \frac{dy/d\theta}{dx/d\theta} \Big|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of  $r = 3 + 2\cos \theta$

$$\text{at } \theta = \frac{\pi}{2} \text{ is } \frac{2}{3}.$$

— OR —

$$y = r\sin \theta = (3 + 2\cos \theta)\sin \theta \Rightarrow \frac{dy}{d\theta} = 3\cos \theta + 2\cos^2 \theta - 2\sin^2 \theta$$

$$x = r\cos \theta = (3 + 2\cos \theta)\cos \theta \Rightarrow \frac{dx}{d\theta} = -3\sin \theta - 4\sin \theta \cos \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \frac{dy/d\theta}{dx/d\theta} \Big|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of  $r = 3 + 2\cos \theta$

$$\text{at } \theta = \frac{\pi}{2} \text{ is } \frac{2}{3}.$$

$$(c) \quad \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = -2\sin \theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin \theta}$$

$$\left. \frac{d\theta}{dt} \right|_{\theta=\pi/3} = 3 \cdot \frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$$3: \begin{cases} 1: \text{constant and limits} \\ 2: \text{integrand} \end{cases}$$

$$3: \begin{cases} 1: \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin \theta + r\cos \theta \\ \quad \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos \theta - r\sin \theta \\ 1: \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin \theta + r\cos \theta}{\frac{dr}{d\theta}\cos \theta - r\sin \theta} \\ 1: \text{answer} \end{cases}$$

$$3: \begin{cases} 1: \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} \\ 1: \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin \theta} \\ 1: \text{answer with units} \end{cases}$$

## 2018 SCORING GUIDELINES

## Question 6

(a) The first four nonzero terms are  $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$ .

The general term is  $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$ .

$$(b) \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})} \cdot \frac{n \cdot 3^n}{(-1)^{n+1} x^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$$\left| \frac{x}{3} \right| < 1 \text{ for } |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for  $f$  is 3.

— OR —

The radius of convergence of the Maclaurin series for  $\ln(1+x)$  is 1, so the series for  $f(x) = x \ln\left(1 + \frac{x}{3}\right)$  converges absolutely for  $\left|\frac{x}{3}\right| < 1$ .

$$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for  $f$  is 3.

When  $x = -3$ , the series is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$ , which diverges by comparison to the harmonic series.

When  $x = 3$ , the series is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$ , which converges by the alternating series test.

The interval of convergence of the Maclaurin series for  $f$  is  $-3 < x \leq 3$ .

(c) By the alternating series error bound, an upper bound for  $|P_4(2) - f(2)|$  is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$

2 :  $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

5 :  $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

— OR —

5 :  $\begin{cases} 1 : \text{radius for } \ln(1+x) \text{ series} \\ 1 : \text{substitutes } \frac{x}{3} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

2 :  $\begin{cases} 1 : \text{uses fifth-degree term as error bound} \\ 1 : \text{answer} \end{cases}$

# AP<sup>®</sup> Calculus BC

## 2016 Scoring Guidelines

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## 2016 SCORING GUIDELINES

## Question 1

|                           |      |      |     |     |     |
|---------------------------|------|------|-----|-----|-----|
| $t$<br>(hours)            | 0    | 1    | 3   | 6   | 8   |
| $R(t)$<br>(liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

Water is pumped into a tank at a rate modeled by  $W(t) = 2000e^{-t^2/20}$  liters per hour for  $0 \leq t \leq 8$ , where  $t$  is measured in hours. Water is removed from the tank at a rate modeled by  $R(t)$  liters per hour, where  $R$  is differentiable and decreasing on  $0 \leq t \leq 8$ . Selected values of  $R(t)$  are shown in the table above. At time  $t = 0$ , there are 50,000 liters of water in the tank.

- (a) Estimate  $R'(2)$ . Show the work that leads to your answer. Indicate units of measure.
- (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- (d) For  $0 \leq t \leq 8$ , is there a time  $t$  when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a)  $R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120$  liters/hr<sup>2</sup>

2 :  $\begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by  $\int_0^8 R(t) dt$ .

$$\begin{aligned} \int_0^8 R(t) dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

3 :  $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

This is an overestimate since  $R$  is a decreasing function.

(c) Total  $\approx 50000 + \int_0^8 W(t) dt - 8050$   
 $= 50000 + 7836.195325 - 8050 \approx 49786$  liters

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d)  $W(0) - R(0) > 0$ ,  $W(8) - R(8) < 0$ , and  $W(t) - R(t)$  is continuous.

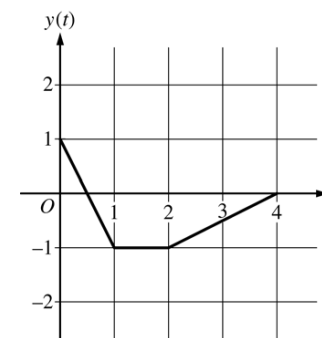
2 :  $\begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

Therefore, the Intermediate Value Theorem guarantees at least one time  $t$ ,  $0 < t < 8$ , for which  $W(t) - R(t) = 0$ , or  $W(t) = R(t)$ .

For this value of  $t$ , the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

## 2016 SCORING GUIDELINES

## Question 2



At time  $t$ , the position of a particle moving in the  $xy$ -plane is given by the parametric functions  $(x(t), y(t))$ , where  $\frac{dx}{dt} = t^2 + \sin(3t^2)$ . The graph of  $y$ , consisting of three line segments, is shown in the figure above. At  $t = 0$ , the particle is at position  $(5, 1)$ .

- (a) Find the position of the particle at  $t = 3$ .
- (b) Find the slope of the line tangent to the path of the particle at  $t = 3$ .
- (c) Find the speed of the particle at  $t = 3$ .
- (d) Find the total distance traveled by the particle from  $t = 0$  to  $t = 2$ .

(a)  $x(3) = x(0) + \int_0^3 x'(t) dt = 5 + 9.377035 = 14.377$

$$y(3) = -\frac{1}{2}$$

The position of the particle at  $t = 3$  is  $(14.377, -0.5)$ .

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) Slope  $= \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$

1 : slope

(c) Speed  $= \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969$  (or 9.968)

2 :  $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

(d) Distance  $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt$   
 $= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2} dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2} dt$   
 $= 2.237871 + 2.112003 = 4.350$  (or 4.349)

3 :  $\begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

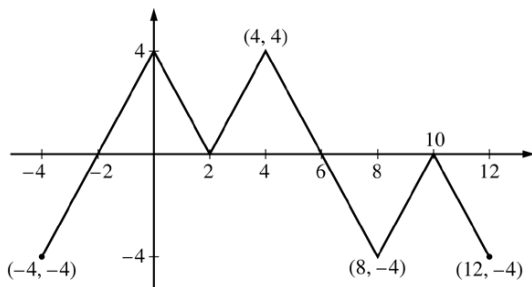
## 2016 SCORING GUIDELINES

## Question 3

The figure above shows the graph of the piecewise-linear function  $f$ . For  $-4 \leq x \leq 12$ , the function  $g$  is defined by

$$g(x) = \int_2^x f(t) dt.$$

- (a) Does  $g$  have a relative minimum, a relative maximum, or neither at  $x = 10$ ? Justify your answer.
- (b) Does the graph of  $g$  have a point of inflection at  $x = 4$ ? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of  $g$  on the interval  $-4 \leq x \leq 12$ . Justify your answers.
- (d) For  $-4 \leq x \leq 12$ , find all intervals for which  $g(x) \leq 0$ .

Graph of  $f$ 

- (a) The function  $g$  has neither a relative minimum nor a relative maximum at  $x = 10$  since  $g'(x) = f(x)$  and  $f(x) \leq 0$  for  $8 \leq x \leq 12$ .
- (b) The graph of  $g$  has a point of inflection at  $x = 4$  since  $g'(x) = f(x)$  is increasing for  $2 \leq x \leq 4$  and decreasing for  $4 \leq x \leq 8$ .
- (c)  $g'(x) = f(x)$  changes sign only at  $x = -2$  and  $x = 6$ .

| $x$ | $g(x)$ |
|-----|--------|
| -4  | -4     |
| -2  | -8     |
| 6   | 8      |
| 12  | -4     |

On the interval  $-4 \leq x \leq 12$ , the absolute minimum value is  $g(-2) = -8$  and the absolute maximum value is  $g(6) = 8$ .

- (d)  $g(x) \leq 0$  for  $-4 \leq x \leq 2$  and  $10 \leq x \leq 12$ .

1 :  $g'(x) = f(x)$  in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 :  $\begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$

2 : intervals

## 2016 SCORING GUIDELINES

## Question 4

Consider the differential equation  $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ .

- (a) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ .
- (b) Let  $y = f(x)$  be the particular solution to the given differential equation whose graph passes through the point  $(-2, 8)$ . Does the graph of  $f$  have a relative minimum, a relative maximum, or neither at the point  $(-2, 8)$ ? Justify your answer.
- (c) Let  $y = g(x)$  be the particular solution to the given differential equation with  $g(-1) = 2$ . Find  $\lim_{x \rightarrow -1} \left( \frac{g(x) - 2}{3(x+1)^2} \right)$ . Show the work that leads to your answer.
- (d) Let  $y = h(x)$  be the particular solution to the given differential equation with  $h(0) = 2$ . Use Euler's method, starting at  $x = 0$  with two steps of equal size, to approximate  $h(1)$ .

(a)  $\frac{d^2y}{dx^2} = 2x - \frac{1}{2} \frac{dy}{dx} = 2x - \frac{1}{2} \left( x^2 - \frac{1}{2}y \right)$

(b)  $\left. \frac{dy}{dx} \right|_{(x,y)=(-2,8)} = (-2)^2 - \frac{1}{2} \cdot 8 = 0$   
 $\left. \frac{d^2y}{dx^2} \right|_{(x,y)=(-2,8)} = 2(-2) - \frac{1}{2} \left( (-2)^2 - \frac{1}{2} \cdot 8 \right) = -4 < 0$

Thus, the graph of  $f$  has a relative maximum at the point  $(-2, 8)$ .

(c)  $\lim_{x \rightarrow -1} (g(x) - 2) = 0$  and  $\lim_{x \rightarrow -1} 3(x+1)^2 = 0$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left( \frac{g(x) - 2}{3(x+1)^2} \right) = \lim_{x \rightarrow -1} \left( \frac{g'(x)}{6(x+1)} \right)$$

$$\lim_{x \rightarrow -1} g'(x) = 0 \text{ and } \lim_{x \rightarrow -1} 6(x+1) = 0$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left( \frac{g'(x)}{6(x+1)} \right) = \lim_{x \rightarrow -1} \left( \frac{g''(x)}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$$

(d)  $h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1) \cdot \frac{1}{2} = \frac{3}{2}$   
 $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{5}{4}$

2 :  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$

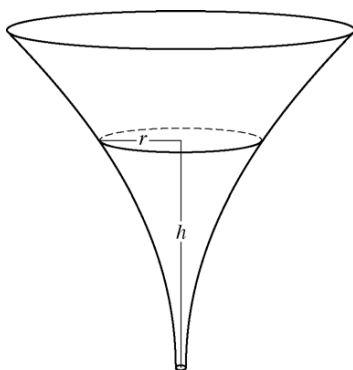
2 : conclusion with justification

3 :  $\begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

2 :  $\begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

## 2016 SCORING GUIDELINES

## Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height  $h$ , the radius of the funnel is given by  $r = \frac{1}{20}(3 + h^2)$ , where  $0 \leq h \leq 10$ . The units of  $r$  and  $h$  are inches.

- Find the average value of the radius of the funnel.
- Find the volume of the funnel.
- The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is  $h = 3$  inches, the radius of the surface of the liquid is decreasing at a rate of  $\frac{1}{5}$  inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\begin{aligned} \text{(a) Average radius} &= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[ 3h + \frac{h^3}{3} \right]_0^{10} \\ &= \frac{1}{200} \left( \left( 30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60} \text{ in} \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_0^{10} \left( \frac{1}{20}(3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh \\ &= \frac{\pi}{400} \left[ 9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} \\ &= \frac{\pi}{400} \left( \left( 90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40} \text{ in}^3 \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(c) } \frac{dr}{dt} &= \frac{1}{20}(2h) \frac{dh}{dt} \\ -\frac{1}{5} &= \frac{3}{10} \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec} \end{aligned}$$

3 :  $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

## 2016 SCORING GUIDELINES

## Question 6

The function  $f$  has a Taylor series about  $x = 1$  that converges to  $f(x)$  for all  $x$  in the interval of convergence.

It is known that  $f(1) = 1$ ,  $f'(1) = -\frac{1}{2}$ , and the  $n$ th derivative of  $f$  at  $x = 1$  is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- Write the first four nonzero terms and the general term of the Taylor series for  $f$  about  $x = 1$ .
- The Taylor series for  $f$  about  $x = 1$  has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- The Taylor series for  $f$  about  $x = 1$  can be used to represent  $f(1.2)$  as an alternating series. Use the first three nonzero terms of the alternating series to approximate  $f(1.2)$ .
- Show that the approximation found in part (c) is within 0.001 of the exact value of  $f(1.2)$ .

$$\begin{aligned} \text{(a) } f(1) &= 1, \quad f'(1) = -\frac{1}{2}, \quad f''(1) = \frac{1}{2^2}, \quad f'''(1) = -\frac{2}{2^3} \\ f(x) &= 1 - \frac{1}{2}(x-1) + \frac{1}{2^2} \cdot \frac{1}{2}(x-1)^2 - \frac{1}{2^3} \cdot \frac{1}{3}(x-1)^3 + \cdots \\ &\quad + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots \end{aligned}$$

4 :  $\begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$

- $R = 2$ . The series converges on the interval  $(-1, 3)$ .

When  $x = -1$ , the series is  $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ .

Since the harmonic series diverges, this series diverges.

When  $x = 3$ , the series is  $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$ .

Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is  $-1 < x \leq 3$ .

$$\text{(c) } f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$$

1 : approximation

- The series for  $f(1.2)$  alternates with terms that decrease in magnitude to 0.

2 :  $\begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$

# AP<sup>®</sup> Calculus BC

## 2015 Scoring Guidelines

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### 2015 SCORING GUIDELINES

#### Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function  $R$ , where  $R(t) = 20\sin\left(\frac{t^2}{35}\right)$  cubic feet per hour,  $t$  is measured in hours, and  $0 \leq t \leq 8$ . The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by  $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$  cubic feet per hour, for  $0 \leq t \leq 8$ . There are 30 cubic feet of water in the pipe at time  $t = 0$ .

- How many cubic feet of rainwater flow into the pipe during the 8-hour time interval  $0 \leq t \leq 8$ ?
- Is the amount of water in the pipe increasing or decreasing at time  $t = 3$  hours? Give a reason for your answer.
- At what time  $t$ ,  $0 \leq t \leq 8$ , is the amount of water in the pipe at a minimum? Justify your answer.
- The pipe can hold 50 cubic feet of water before overflowing. For  $t > 8$ , water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time  $w$  when the pipe will begin to overflow.

(a)  $\int_0^8 R(t) dt = 76.570$

2 :  $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b)  $R(3) - D(3) = -0.313632 < 0$

Since  $R(3) < D(3)$ , the amount of water in the pipe is decreasing at time  $t = 3$  hours.

2 :  $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time  $t$ ,  $0 \leq t \leq 8$ , is  $30 + \int_0^t [R(x) - D(x)] dx$ .

3 :  $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$$

| $t$      | Amount of water in the pipe |
|----------|-----------------------------|
| 0        | 30                          |
| 3.271658 | 27.964561                   |
| 8        | 48.543686                   |

The amount of water in the pipe is a minimum at time  $t = 3.272$  (or 3.271) hours.

(d)  $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 2

At time  $t \geq 0$ , a particle moving along a curve in the  $xy$ -plane has position  $(x(t), y(t))$  with velocity vector  $v(t) = (\cos(t^2), e^{0.5t})$ . At  $t = 1$ , the particle is at the point  $(3, 5)$ .

- (a) Find the  $x$ -coordinate of the position of the particle at time  $t = 2$ .
- (b) For  $0 < t < 1$ , there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time  $t = 0$  to time  $t = 1$ .

(a)  $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$  (or 2.556)

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b)  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

2 :  $\begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$

(c) Speed =  $\sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

2 :  $\begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$

(d) Distance =  $\int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$  (or 1.594)

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 3

|                               |   |     |     |      |     |
|-------------------------------|---|-----|-----|------|-----|
| $t$<br>(minutes)              | 0 | 12  | 20  | 24   | 40  |
| $v(t)$<br>(meters per minute) | 0 | 200 | 240 | -220 | 150 |

Johanna jogs along a straight path. For  $0 \leq t \leq 40$ , Johanna's velocity is given by a differentiable function  $v$ . Selected values of  $v(t)$ , where  $t$  is measured in minutes and  $v(t)$  is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of  $v'(16)$ .

- (b) Using correct units, explain the meaning of the definite integral  $\int_0^{40} |v(t)| dt$  in the context of the problem.

Approximate the value of  $\int_0^{40} |v(t)| dt$  using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For  $0 \leq t \leq 10$ , Bob's velocity is modeled by  $B(t) = t^3 - 6t^2 + 300$ , where  $t$  is measured in minutes and  $B(t)$  is measured in meters per minute.

Find Bob's acceleration at time  $t = 5$ .

- (d) Based on the model  $B$  from part (c), find Bob's average velocity during the interval  $0 \leq t \leq 10$ .

(a)  $v'(16) \approx \frac{240 - 200}{20 - 12} = 5$  meters/min<sup>2</sup>

1 : approximation

- (b)  $\int_0^{40} |v(t)| dt$  is the total distance Johanna jogs, in meters, over the time interval  $0 \leq t \leq 40$  minutes.

3 :  $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is  $B'(t) = 3t^2 - 12t$ .

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 :  $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

(d) Avg vel =  $\frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$= \frac{1}{10} \left[ \frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$$

$$= \frac{1}{10} \left[ \frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min}$$

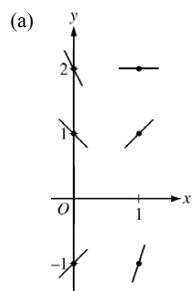
3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 4

Consider the differential equation  $\frac{dy}{dx} = 2x - y$ .

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let  $y = f(x)$  be the particular solution to the differential equation with the initial condition  $f(2) = 3$ . Does  $f$  have a relative minimum, a relative maximum, or neither at  $x = 2$ ? Justify your answer.
- (d) Find the values of the constants  $m$  and  $b$  for which  $y = mx + b$  is a solution to the differential equation.



- 2 :  $\begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b)  $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II,  $x < 0$  and  $y > 0$ , so  $2 - 2x + y > 0$ .  
Therefore, all solution curves are concave up in Quadrant II.

- 2 :  $\begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c)  $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore,  $f$  has neither a relative minimum nor a relative maximum at  $x = 2$ .

- 2 :  $\begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d)  $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

- 3 :  $\begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 5

Consider the function  $f(x) = \frac{1}{x^2 - kx}$ , where  $k$  is a nonzero constant. The derivative of  $f$  is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let  $k = 3$ , so that  $f(x) = \frac{1}{x^2 - 3x}$ . Write an equation for the line tangent to the graph of  $f$  at the point whose  $x$ -coordinate is 4.
- (b) Let  $k = 4$ , so that  $f(x) = \frac{1}{x^2 - 4x}$ . Determine whether  $f$  has a relative minimum, a relative maximum, or neither at  $x = 2$ . Justify your answer.
- (c) Find the value of  $k$  for which  $f$  has a critical point at  $x = -5$ .
- (d) Let  $k = 6$ , so that  $f(x) = \frac{1}{x^2 - 6x}$ . Find the partial fraction decomposition for the function  $f$ .

Find  $\int f(x) dx$ .

(a)  $f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4}$        $f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$

An equation for the line tangent to the graph of  $f$  at the point whose  $x$ -coordinate is 4 is  $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$ .

- 2 :  $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

(b)  $f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2}$        $f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$

$f'(x)$  changes sign from positive to negative at  $x = 2$ .  
Therefore,  $f$  has a relative maximum at  $x = 2$ .

- 2 :  $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

(c)  $f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$

- 1 : answer

(d)  $\frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left( \frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

- 4 :  $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

## 2015 SCORING GUIDELINES

## Question 6

The Maclaurin series for a function  $f$  is given by  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$  and converges to  $f(x)$  for  $|x| < R$ , where  $R$  is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find  $R$ .
- (b) Write the first four nonzero terms of the Maclaurin series for  $f'$ , the derivative of  $f$ . Express  $f'$  as a rational function for  $|x| < R$ .
- (c) Write the first four nonzero terms of the Maclaurin series for  $e^x$ . Use the Maclaurin series for  $e^x$  to write the third-degree Taylor polynomial for  $g(x) = e^x f(x)$  about  $x = 0$ .

- (a) Let  $a_n$  be the  $n$ th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is  $R = \frac{1}{3}$ .

- (b) The first four nonzero terms of the Maclaurin series for  $f'$  are

$$1 - 3x + 9x^2 - 27x^3.$$

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for  $e^x$  are

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The product of the Maclaurin series for  $e^x$  and the Maclaurin series for  $f$  is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x - \frac{3}{2}x^2 + 3x^3 - \dots\right) \\ = x - \frac{1}{2}x^2 + 2x^3 + \dots$$

The third-degree Taylor polynomial for  $g(x) = e^x f(x)$

about  $x = 0$  is  $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$ .

1 : sets up ratio  
3 :  $\begin{cases} 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

3 :  $\begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$

3 :  $\begin{cases} 1 : \text{first four nonzero terms} \\ \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$