

Part A (AB or BC): Graphing calculator required**Question 1****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f , where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

Model Solution	Scoring
(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t) dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t) dt$.	
$\int_{60}^{135} f(t) dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.	Interpretation with units 1 point
$\int_{60}^{135} f(t) dt$ $\approx f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120)$ $= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$	Form of Riemann sum 1 point Answer 1 point

Scoring notes:

- To earn the first point the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.

- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30) + 0.1(30) + 0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only 8.25 does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9$, or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

Total for part (a) 3 points

- (b) Must there exist a value of c , for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

f is differentiable. $\Rightarrow f$ is continuous on $[60, 120]$.	$f(120) - f(60) = 0$ 1 point
$\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$	Answer with justification 1 point
By the Mean Value Theorem, there must exist a c , for $60 < c < 120$, such that $f'(c) = 0$.	

Scoring notes:

- To earn the first point a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must:
 - have earned the first point,
 - state that f is continuous because f is differentiable (or equivalent), and
 - answer “yes” in some way.
- A response may reference either the Mean Value Theorem or Rolle’s Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

Total for part (b) 2 points

- (c) The rate of flow of gasoline, in gallons per second, can also be modeled by

$g(t) = \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \leq t \leq 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \leq t \leq 150$. Show the setup for your calculations.

$\frac{1}{150 - 0} \int_0^{150} g(t) dt$	Average value formula 1 point
$= 0.0959967$	Answer 1 point

The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).

Scoring notes:

- The exact value of $\frac{1}{150} \int_0^{150} g(t) dt$ is $\frac{12}{125} \sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t) dt = 14.399504$ so the average rate is 0.0959967.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication (e.g., $\int_0^{150} g(t) dt = \frac{14.399504}{150} = 0.0959967$) earns 1 out of 2 points.
- A response of $\int_0^{150} g(t) dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150} \int_0^{150} g(t) dt = 0.149981$ or 0.002618.

Total for part (c) 2 points

- (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

$g'(140) \approx -0.004908$ $g'(140) = -0.005$ (or -0.004)	$g'(140)$ 1 point
The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.	Interpretation 1 point

Scoring notes:

- The exact value of $g'(140)$ is $\frac{1}{500} \cos\left(\frac{49}{36}\right) - \frac{49}{9000} \sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point a response must present some numerical value for $g'(140)$.
- To earn the second point the interpretation must include “the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]” and “at $t = 140$ ” (or equivalent).
- An interpretation of “decreasing at a rate of -0.005 ” or “increasing at a rate of 0.005” does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or 0.00187.

Total for part (d) 2 points**Total for question 1 9 points**

2019

AP[®] Calculus BC

Scoring Guidelines

2019 SCORING GUIDELINES

Question 1

(a) $\int_0^5 E(t) dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

(b) $\frac{1}{5-0} \int_0^5 L(t) dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

- (c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$$E(t) - L(t) = 0 \Rightarrow t = 6.20356$$

$E(t) - L(t) > 0$ for $0 \leq t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \leq 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$$A(t) = \int_0^t (E(s) - L(s)) ds$$

$$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

2019 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = 3.534292$

The area of S is 3.534.

(b) $\frac{1}{\sqrt{\pi}-0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \leq \theta \leq \sqrt{\pi}$ is 1.580 (or 1.579).

(c) $\tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$

$$\frac{1}{2} \int_0^{\tan^{-1} m} (r(\theta))^2 d\theta = \frac{1}{2} \left(\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta \right)$$

- (d) As $k \rightarrow \infty$, the circle $r = k \cos \theta$ grows to enclose all points to the right of the y -axis.

$$\begin{aligned} \lim_{k \rightarrow \infty} A(k) &= \frac{1}{2} \int_0^{\pi/2} (r(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} (3\sqrt{\theta} \sin(\theta^2))^2 d\theta = 3.324 \end{aligned}$$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \text{applied to } m \text{ as limit of} \\ \text{integration} \\ 1 : \text{equation} \end{cases}$

2 : $\begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

2019 SCORING GUIDELINES

Question 3

$$\begin{aligned} \text{(a)} \quad \int_{-6}^5 f(x) \, dx &= \int_{-6}^{-2} f(x) \, dx + \int_{-2}^5 f(x) \, dx \\ &\Rightarrow 7 = \int_{-6}^{-2} f(x) \, dx + 2 + \left(9 - \frac{9\pi}{4}\right) \\ &\Rightarrow \int_{-6}^{-2} f(x) \, dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_3^5 (2f'(x) + 4) \, dx &= 2 \int_3^5 f'(x) \, dx + \int_3^5 4 \, dx \\ &= 2(f(5) - f(3)) + 4(5 - 3) \\ &= 2(0 - (3 - \sqrt{5})) + 8 \\ &= 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5} \end{aligned}$$

— OR —

$$\begin{aligned} \int_3^5 (2f'(x) + 4) \, dx &= [2f(x) + 4x]_{x=3}^{x=5} \\ &= (2f(5) + 20) - (2f(3) + 12) \\ &= (2 \cdot 0 + 20) - (2(3 - \sqrt{5}) + 12) \\ &= 2 + 2\sqrt{5} \end{aligned}$$

$$\text{(c)} \quad g'(x) = f(x) = 0 \Rightarrow x = -1, x = \frac{1}{2}, x = 5$$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$$\begin{aligned} \text{(d)} \quad \lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} &= \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \\ &= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}} \end{aligned}$$

$$3 : \begin{cases} 1 : \int_{-6}^5 f(x) \, dx = \int_{-6}^{-2} f(x) \, dx + \int_{-2}^5 f(x) \, dx \\ 1 : \int_{-2}^5 f(x) \, dx \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$$

1 : answer

2019 SCORING GUIDELINES

Question 4

$$\begin{aligned} \text{(a)} \quad V &= \pi r^2 h = \pi(1)^2 h = \pi h \\ \frac{dV}{dt} \Big|_{h=4} &= \pi \frac{dh}{dt} \Big|_{h=4} = \pi \left(-\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d^2 h}{dt^2} &= -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200} \\ \text{Because } \frac{d^2 h}{dt^2} &= \frac{1}{200} > 0 \text{ for } h > 0, \text{ the rate of change of the} \\ \text{height is increasing when the height of the water is 3 feet.} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{dh}{\sqrt{h}} &= -\frac{1}{10} dt \\ \int \frac{dh}{\sqrt{h}} &= \int -\frac{1}{10} dt \\ 2\sqrt{h} &= -\frac{1}{10} t + C \\ 2\sqrt{5} &= -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5} \\ 2\sqrt{h} &= -\frac{1}{10} t + 2\sqrt{5} \\ h(t) &= \left(-\frac{1}{20} t + \sqrt{5} \right)^2 \end{aligned}$$

$$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration

2019 SCORING GUIDELINES

Question 5

$$(a) f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2}$$

$$f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

$$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$$

$$(b) \frac{1}{x^2-2x-8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$\Rightarrow 1 = A(x+2) + B(x-4)$$

$$\Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}$$

$$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$$

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left(\frac{1}{x-4} - \frac{1}{x+2} \right) dx \\ &= \left[\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1} \\ &= \left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \end{aligned}$$

$$\begin{aligned} (c) \int_0^2 \frac{1}{x^2-2x+1} dx &= \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right) \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right) \end{aligned}$$

Because $\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$$

2019 SCORING GUIDELINES

Question 6

$$(a) f(0) = 3 \text{ and } f'(0) = -2$$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-2}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$$

$$(b) \text{ The first three nonzero terms of the Maclaurin series for } e^x \text{ are}$$

$$1 + x + \frac{1}{2!}x^2.$$

$$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$$

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} 3 \left(1 + x + \frac{1}{2!}x^2 \right) - 2x(1+x) + \frac{3}{2}x^2(1) \\ = 3 + (3-2)x + \left(\frac{3}{2} - 2 + \frac{3}{2} \right)x^2 \\ = 3 + x + x^2. \end{aligned}$$

$$(c) h(1) = \int_0^1 f(t) dt$$

$$\begin{aligned} &\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3 \right) dt \\ &= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4 \right]_{t=0}^{t=1} \\ &= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48} \end{aligned}$$

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

$$(d) \text{ The alternating series error bound is the absolute value of the first omitted term of the series for } h(1).$$

$$\int_0^1 \left(\frac{54}{4!}t^4 \right) dt = \left[\frac{9}{20}t^5 \right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \leq \left| \frac{9}{20} \right| = 0.45$$

$$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$$

AP Calculus BC

Scoring Guidelines

2018 SCORING GUIDELINES

Question 1

(a) $\int_0^{300} r(t) \, dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval $0 \leq t \leq 300$.

(b) $20 + \int_0^{300} (r(t) - 0.7) \, dt = 20 + \int_0^{300} r(t) \, dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time $t = 300$.

(c) Based on part (b), the number of people in line at time $t = 300$ is 80.

The first time t that there are no people in line is

$$300 + \frac{80}{0.7} = 414.286 \text{ (or 414.285) seconds.}$$

(d) The total number of people in line at time t , $0 \leq t \leq 300$, is modeled by $20 + \int_0^t r(x) \, dx - 0.7t$.

$$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

1 : answer

4 : $\begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

2018 SCORING GUIDELINES

Question 2

(a) $p'(25) = -1.179$

At a depth of 25 meters, the density of plankton cells is changing at a rate of -1.179 million cells per cubic meter per meter.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{meaning with units} \end{cases}$$

(b) $\int_0^{30} 3p(h) \, dh = 1675.414936$

There are 1675 million plankton cells in the column of water between $h = 0$ and $h = 30$ meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(c) $\int_{30}^K 3f(h) \, dh$ represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of K meters.

The number of plankton cells, in millions, in the entire column of water is given by $\int_0^{30} 3p(h) \, dh + \int_{30}^K 3f(h) \, dh$.

Because $0 \leq f(h) \leq u(h)$ for all $h \geq 30$,

$$3 \int_{30}^K f(h) \, dh \leq 3 \int_{30}^K u(h) \, dh \leq 3 \int_{30}^{\infty} u(h) \, dh = 3 \cdot 105 = 315.$$

The total number of plankton cells in the column of water is bounded by $1675.415 + 315 = 1990.415 \leq 2000$ million.

$$3 : \begin{cases} 1 : \text{integral expression} \\ 1 : \text{compares improper integral} \\ 1 : \text{explanation} \end{cases}$$

(d) $\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = 757.455862$

The total distance traveled by the boat over the time interval $0 \leq t \leq 1$ is 757.456 (or 757.455) meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{total distance} \end{cases}$$

2018 SCORING GUIDELINES

Question 3

(a) $f(-5) = f(1) + \int_1^{-5} g(x) \, dx = f(1) - \int_{-5}^1 g(x) \, dx$
 $= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b) $\int_1^6 g(x) \, dx = \int_1^3 g(x) \, dx + \int_3^6 g(x) \, dx$
 $= \int_1^3 2 \, dx + \int_3^6 2(x-4)^2 \, dx$
 $= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$

$$3 : \begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f''(x) = g(x)$ is increasing on those intervals.

$$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$$

(d) The graph of f has a point of inflection at $x = 4$ because $f''(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$$

2018 SCORING GUIDELINES

Question 4

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$.

By the Mean Value Theorem, there exists a value c , $3 < c < 5$, such that $H'(c) = 2$.

(c) The average height of the tree over the time interval $2 \leq t \leq 10$ is given by $\frac{1}{10 - 2} \int_2^{10} H(t) \, dt$.

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) \, dt &\approx \frac{1}{8} \left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval $2 \leq t \leq 10$ is $\frac{263}{32}$ meters.

$$(d) \quad G(x) = 50 \Rightarrow x = 1$$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

2 : $\begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

2 : $\begin{cases} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$

2 : $\begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$

3 : $\begin{cases} 2 : \frac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0] if no chain rule

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Question 5

$$(a) \quad \text{Area} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} (4^2 - (3 + 2\cos \theta)^2) \, d\theta$$

$$(b) \quad \frac{dr}{d\theta} = -2\sin \theta \Rightarrow \left. \frac{dr}{d\theta} \right|_{\theta=\pi/2} = -2$$

$$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$$

$$y = r\sin \theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r\cos \theta$$

$$x = r\cos \theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r\sin \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \frac{dy/d\theta}{dx/d\theta} \Big|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos \theta$

at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

— OR —

$$y = r\sin \theta = (3 + 2\cos \theta)\sin \theta \Rightarrow \frac{dy}{d\theta} = 3\cos \theta + 2\cos^2 \theta - 2\sin^2 \theta$$

$$x = r\cos \theta = (3 + 2\cos \theta)\cos \theta \Rightarrow \frac{dx}{d\theta} = -3\sin \theta - 4\sin \theta \cos \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \frac{dy/d\theta}{dx/d\theta} \Big|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos \theta$

at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

$$(c) \quad \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = -2\sin \theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin \theta}$$

$$\left. \frac{d\theta}{dt} \right|_{\theta=\pi/3} = 3 \cdot \frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

3 : $\begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$

3 : $\begin{cases} 1 : \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r\cos \theta \\ \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r\sin \theta \\ 1 : \frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r\cos \theta}{\frac{dr}{d\theta} \cos \theta - r\sin \theta} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} \\ 1 : \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin \theta} \\ 1 : \text{answer with units} \end{cases}$

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Question 6

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

(b)
$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$\left| \frac{x}{3} \right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x \ln\left(1 + \frac{x}{3}\right)$ converges absolutely for $\left|\frac{x}{3}\right| < 1$.

$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which

diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which

converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

5 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

— OR —

5 : $\begin{cases} 1 : \text{radius for } \ln(1+x) \text{ series} \\ 1 : \text{substitutes } \frac{x}{3} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

2 : $\begin{cases} 1 : \text{uses fifth-degree term as error bound} \\ 1 : \text{answer} \end{cases}$