

AP[®] Calculus BC

Scoring Guidelines

2019 SCORING GUIDELINES

Question 1

(a) $\int_0^5 E(t) dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

(b) $\frac{1}{5-0} \int_0^5 L(t) dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

- (c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$$E(t) - L(t) = 0 \Rightarrow t = 6.20356$$

$E(t) - L(t) > 0$ for $0 \leq t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \leq 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$$A(t) = \int_0^t (E(s) - L(s)) ds$$

$$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

2019 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = 3.534292$

The area of S is 3.534.

(b) $\frac{1}{\sqrt{\pi}-0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \leq \theta \leq \sqrt{\pi}$ is 1.580 (or 1.579).

(c) $\tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$

$$\frac{1}{2} \int_0^{\tan^{-1} m} (r(\theta))^2 d\theta = \frac{1}{2} \left(\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta \right)$$

(d) As $k \rightarrow \infty$, the circle $r = k \cos \theta$ grows to enclose all points to the right of the y -axis.

$$\begin{aligned} \lim_{k \rightarrow \infty} A(k) &= \frac{1}{2} \int_0^{\pi/2} (r(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} (3\sqrt{\theta} \sin(\theta^2))^2 d\theta = 3.324 \end{aligned}$$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \text{applied to } m \text{ as limit of} \\ \text{integration} \\ 1 : \text{equation} \end{cases}$

2 : $\begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

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Question 3

(a) $\int_{-6}^5 f(x) dx = \int_{-6}^{-2} f(x) dx + \int_{-2}^5 f(x) dx$

$$\Rightarrow 7 = \int_{-6}^{-2} f(x) dx + 2 + \left(9 - \frac{9\pi}{4}\right)$$

$$\Rightarrow \int_{-6}^{-2} f(x) dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$$

(b) $\begin{aligned} \int_3^5 (2f'(x) + 4) dx &= 2 \int_3^5 f'(x) dx + \int_3^5 4 dx \\ &= 2(f(5) - f(3)) + 4(5 - 3) \\ &= 2(0 - (3 - \sqrt{5})) + 8 \\ &= 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5} \end{aligned}$

— OR —

$$\begin{aligned} \int_3^5 (2f'(x) + 4) dx &= [2f(x) + 4x]_{x=3}^{x=5} \\ &= (2f(5) + 20) - (2f(3) + 12) \\ &= (2 \cdot 0 + 20) - (2(3 - \sqrt{5}) + 12) \\ &= 2 + 2\sqrt{5} \end{aligned}$$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1, x = \frac{1}{2}, x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

(d) $\begin{aligned} \lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} &= \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \\ &= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}} \end{aligned}$

3 : $\begin{cases} 1 : \int_{-6}^5 f(x) dx = \int_{-6}^{-2} f(x) dx + \int_{-2}^5 f(x) dx \\ 1 : \int_{-2}^5 f(x) dx \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$

1 : answer

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Question 4

(a) $V = \pi r^2 h = \pi(1)^2 h = \pi h$

$$\left. \frac{dV}{dt} \right|_{h=4} = \pi \left. \frac{dh}{dt} \right|_{h=4} = \pi \left(-\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

2 : $\begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $\frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$

Because $\frac{d^2 h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

3 : $\begin{cases} 1 : \frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5} \right)^2$$

4 : $\begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration

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Question 5

(a) $f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2}$

$$f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

3 : $\begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{x^2-2x-8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$

$$\Rightarrow 1 = A(x+2) + B(x-4)$$

$$\Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}$$

3 : $\begin{cases} 1 : \text{partial fraction} \\ \text{decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left(\frac{1}{6} \frac{1}{x-4} - \frac{1}{6} \frac{1}{x+2} \right) dx \\ &= \left[\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1} \\ &= \left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_0^2 \frac{1}{x^2-2x+1} dx &= \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right) \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right) \end{aligned}$$

Because $\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

3 : $\begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

2019 SCORING GUIDELINES

Question 6

- (a)
- $f(0) = 3$
- and
- $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-2}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

- (b) The first three nonzero terms of the Maclaurin series for
- e^x
- are

$$1 + x + \frac{1}{2!}x^2.$$

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} 3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) \\ = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 \\ = 3 + x + x^2. \end{aligned}$$

- (c)
- $h(1) = \int_0^1 f(t) dt$

$$\begin{aligned} &\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt \\ &= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1} \\ &= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48} \end{aligned}$$

- (d) The alternating series error bound is the absolute value of the first omitted term of the series for
- $h(1)$
- .

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \leq \left|\frac{9}{20}\right| = 0.45$$

$$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$$

$$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$$

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : \text{uses fourth-degree term} \\ \quad \text{of Maclaurin series for } f \\ 1 : \text{uses first omitted term} \\ \quad \text{of series for } h(1) \\ 1 : \text{error bound} \end{cases}$$

2017

AP Calculus BC

Scoring Guidelines

2017 SCORING GUIDELINES

Question 1

	1 : units in parts (a), (c), and (d)
(a) Volume = $\int_0^{10} A(h) \, dh$ $\approx (2 - 0) \cdot A(0) + (5 - 2) \cdot A(2) + (10 - 5) \cdot A(5)$ $= 2 \cdot 50.3 + 3 \cdot 14.4 + 5 \cdot 6.5$ $= 176.3 \text{ cubic feet}$	2 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h) \, dh = 101.325338$ The volume is 101.325 cubic feet.	2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x) \, dx$. $\left. \frac{dV}{dt} \right _{h=5} = \left[\frac{dV}{dh} \cdot \frac{dh}{dt} \right]_{h=5}$ $= \left[f(h) \cdot \frac{dh}{dt} \right]_{h=5}$ $= f(5) \cdot 0.26 = 1.694419$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	3 : $\begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

2017 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2} \int_0^{\pi/2} (f(\theta))^2 \, d\theta = 0.648414$ The area of R is 0.648.	2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $\int_0^k ((g(\theta))^2 - (f(\theta))^2) \, d\theta = \frac{1}{2} \int_0^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) \, d\theta$ — OR — $\int_0^k ((g(\theta))^2 - (f(\theta))^2) \, d\theta = \int_k^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) \, d\theta$	2 : $\begin{cases} 1 : \text{integral expression} \\ \text{for one region} \\ 1 : \text{equation} \end{cases}$
(c) $w(\theta) = g(\theta) - f(\theta)$ $w_A = \frac{\int_0^{\pi/2} w(\theta) \, d\theta}{\frac{\pi}{2} - 0} = 0.485446$ The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.	3 : $\begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$
(d) $w(\theta) = w_A$ for $0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \theta = 0.517688$ $w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517). $w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.	2 : $\begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

2017 SCORING GUIDELINES

Question 3

$$(a) f(-6) = f(-2) + \int_{-2}^{-6} f'(x) dx = 7 - \int_{-6}^{-2} f'(x) dx = 7 - 4 = 3$$

$$f(5) = f(-2) + \int_{-2}^5 f'(x) dx = 7 - 2\pi + 3 = 10 - 2\pi$$

$$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$$

$$(b) f'(x) > 0 \text{ on the intervals } [-6, -2) \text{ and } (2, 5).$$

Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

$$(c) \text{ The absolute minimum will occur at a critical point where } f'(x) = 0 \text{ or at an endpoint.}$$

$$f'(x) = 0 \Rightarrow x = -2, x = 2$$

$$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$$(d) f''(-5) = \frac{2-0}{-6-(-2)} = -\frac{1}{2}$$

$$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad \text{with explanation} \end{cases}$$

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2 \text{ and } \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1$$

$f''(3)$ does not exist because

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3}.$$

2017 SCORING GUIDELINES

Question 4

$$(a) H'(0) = -\frac{1}{4}(91 - 27) = -16$$

$$H(0) = 91$$

An equation for the tangent line is $y = 91 - 16t$.

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$(b) \frac{d^2H}{dt^2} = -\frac{1}{4} \frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$$

1 : underestimate with reason

$$H > 27 \text{ for } t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0 \text{ for } t > 0$$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

$$(c) \frac{dG}{(G - 27)^{2/3}} = -dt$$

$$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1) dt$$

$$3(G - 27)^{1/3} = -t + C$$

$$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$$

$$3(G - 27)^{1/3} = 12 - t$$

$$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3 \text{ for } 0 \leq t < 10$$

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

The internal temperature of the potato at time $t = 3$ minutes is

$$27 + \left(\frac{12 - 3}{3}\right)^3 = 54 \text{ degrees Celsius.}$$

2017 SCORING GUIDELINES

Question 5

$$(a) f'(x) = \frac{-3(4x-7)}{(2x^2-7x+5)^2}$$

$$f'(3) = \frac{(-3)(5)}{(18-21+5)^2} = -\frac{15}{4}$$

$$(b) f'(x) = \frac{-3(4x-7)}{(2x^2-7x+5)^2} = 0 \Rightarrow x = \frac{7}{4}$$

The only critical point in the interval $1 < x < 2.5$ has x -coordinate $\frac{7}{4}$.

f' changes sign from positive to negative at $x = \frac{7}{4}$.

Therefore, f has a relative maximum at $x = \frac{7}{4}$.

$$(c) \int_5^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_5^b \frac{3}{2x^2-7x+5} dx = \lim_{b \rightarrow \infty} \int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1} \right) dx$$

$$= \lim_{b \rightarrow \infty} \left[\ln(2x-5) - \ln(x-1) \right]_5^b = \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2x-5}{x-1} \right) \right]_5^b$$

$$= \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2b-5}{b-1} \right) - \ln \left(\frac{5}{4} \right) \right] = \ln 2 - \ln \left(\frac{5}{4} \right) = \ln \left(\frac{8}{5} \right)$$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\int_5^\infty \frac{3}{2x^2-7x+5} dx$ converges.

— OR —

$$\frac{3}{2n^2-7n+5} > 0 \text{ and } \frac{1}{n^2} > 0 \text{ for } n \geq 5.$$

$$\text{Since } \lim_{n \rightarrow \infty} \frac{\frac{3}{2n^2-7n+5}}{\frac{1}{n^2}} = \frac{3}{2} \text{ and the series } \sum_{n=5}^\infty \frac{1}{n^2} \text{ converges,}$$

the series $\sum_{n=5}^\infty \frac{3}{2n^2-7n+5}$ converges by the limit comparison test.

2 : $f'(3)$

2 : $\begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \text{with justification} \end{cases}$

3 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

2 : answer with conditions

2017 SCORING GUIDELINES

Question 6

$$(a) f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = -1(1) = -1$$

$$f'''(0) = -2(-1) = 2$$

$$f^{(4)}(0) = -3(2) = -6$$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

$$(b) \text{ For } x = 1, \text{ the Maclaurin series becomes } \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}.$$

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

$$(c) \int_0^x f(t) dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt$$

$$= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$$

$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded

by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

$$\text{Thus, } \left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}.$$

3 : $\begin{cases} 1 : f''(0), f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$

2 : converges conditionally with reason

3 : $\begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

1 : error bound

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2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t) dt = 76.570$

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$

Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

2 : $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t , $0 \leq t \leq 8$, is $30 + \int_0^t [R(x) - D(x)] dx$.

3 : $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

(d) $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

2015 SCORING GUIDELINES

Question 2

At time $t \geq 0$, a particle moving along a curve in the xy -plane has position $(x(t), y(t))$ with velocity vector $\mathbf{v}(t) = (\cos(t^2), e^{0.5t})$. At $t = 1$, the particle is at the point $(3, 5)$.

- (a) Find the x -coordinate of the position of the particle at time $t = 2$.
- (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

(a) $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$ (or 2.556)

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

2 : $\begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$

(c) Speed = $\sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

2 : $\begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$

(d) Distance = $\int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$ (or 1.594)

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	-220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of $v'(16)$.

- (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

- (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

(a) $v'(16) \approx \frac{240 - 200}{20 - 12} = 5$ meters/min²

1 : approximation

- (b) $\int_0^{40} |v(t)| dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \leq t \leq 40$ minutes.

3 : $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 : $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

(d) Avg vel = $\frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$= \frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10}$$

$$= \frac{1}{10} \left[\frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min}$$

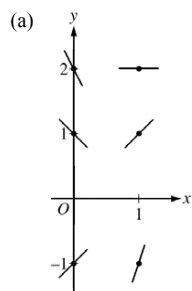
3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.



- 2 : $\begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

- 2 : $\begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

- 2 : $\begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

- 3 : $\begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 5

Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x -coordinate is 4.
- (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.
- (c) Find the value of k for which f has a critical point at $x = -5$.
- (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f .

Find $\int f(x) dx$.

(a) $f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4}$ $f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$

An equation for the line tangent to the graph of f at the point whose x -coordinate is 4 is $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$.

- 2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

(b) $f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2}$ $f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

- 2 : $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

(c) $f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$

- 1 : answer

(d) $\frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

- 4 : $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

2015 SCORING GUIDELINES

Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find R .
- (b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.
- (c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

- (a) Let a_n be the n th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

- (b) The first four nonzero terms of the Maclaurin series for f' are

$$1 - 3x + 9x^2 - 27x^3.$$

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for e^x are

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\begin{aligned} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x - \frac{3}{2}x^2 + 3x^3 - \dots\right) \\ = x - \frac{1}{2}x^2 + 2x^3 + \dots \end{aligned}$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$

about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

1 : sets up ratio
1 : computes limit of ratio
1 : determines radius of convergence

2 : first four nonzero terms
1 : rational function

1 : first four nonzero terms
of the Maclaurin series for e^x
2 : Taylor polynomial