

**2018****AP<sup>®</sup>** **CollegeBoard**

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# AP Calculus BC

## Scoring Guidelines

## 2018 SCORING GUIDELINES

## Question 1

(a)  $\int_0^{300} r(t) dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval  $0 \leq t \leq 300$ .

(b)  $20 + \int_0^{300} (r(t) - 0.7) dt = 20 + \int_0^{300} r(t) dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time  $t = 300$ .

(c) Based on part (b), the number of people in line at time  $t = 300$  is 80.

The first time  $t$  that there are no people in line is

$$300 + \frac{80}{0.7} = 414.286 \text{ (or 414.285) seconds.}$$

(d) The total number of people in line at time  $t$ ,  $0 \leq t \leq 300$ , is modeled by

$$20 + \int_0^t r(x) dx - 0.7t.$$

$$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$$

| $t$   | People in line for escalator |
|-------|------------------------------|
| 0     | 20                           |
| $t_1$ | 3.803                        |
| $t_2$ | 158.070                      |
| 300   | 80                           |

The number of people in line is a minimum at time  $t = 33.013$  seconds, when there are 4 people in line.

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

2 :  $\begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

1 : answer

4 :  $\begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

## 2018 SCORING GUIDELINES

## Question 2

(a)  $p'(25) = -1.179$

At a depth of 25 meters, the density of plankton cells is changing at a rate of  $-1.179$  million cells per cubic meter per meter.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{meaning with units} \end{cases}$$

(b)  $\int_0^{30} 3p(h) \, dh = 1675.414936$

There are 1675 million plankton cells in the column of water between  $h = 0$  and  $h = 30$  meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(c)  $\int_{30}^K 3f(h) \, dh$  represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of  $K$  meters.

The number of plankton cells, in millions, in the entire column of water is given by  $\int_0^{30} 3p(h) \, dh + \int_{30}^K 3f(h) \, dh$ .

Because  $0 \leq f(h) \leq u(h)$  for all  $h \geq 30$ ,

$$3 \int_{30}^K f(h) \, dh \leq 3 \int_{30}^K u(h) \, dh \leq 3 \int_{30}^{\infty} u(h) \, dh = 3 \cdot 105 = 315.$$

The total number of plankton cells in the column of water is bounded by  $1675.415 + 315 = 1990.415 \leq 2000$  million.

$$3 : \begin{cases} 1 : \text{integral expression} \\ 1 : \text{compares improper integral} \\ 1 : \text{explanation} \end{cases}$$

(d)  $\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = 757.455862$

The total distance traveled by the boat over the time interval  $0 \leq t \leq 1$  is 757.456 (or 757.455) meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{total distance} \end{cases}$$

# 2018 SCORING GUIDELINES

## Question 3

$$\begin{aligned} \text{(a)} \quad f(-5) &= f(1) + \int_1^{-5} g(x) \, dx = f(1) - \int_{-5}^1 g(x) \, dx \\ &= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2} \end{aligned}$$

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b)} \quad \int_1^6 g(x) \, dx &= \int_1^3 g(x) \, dx + \int_3^6 g(x) \, dx \\ &= \int_1^3 2 \, dx + \int_3^6 2(x-4)^2 \, dx \\ &= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10 \end{aligned}$$

3 :  $\begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$

(c) The graph of  $f$  is increasing and concave up on  $0 < x < 1$  and  $4 < x < 6$  because  $f'(x) = g(x) > 0$  and  $f''(x) = g'(x)$  is increasing on those intervals.

2 :  $\begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(d) The graph of  $f$  has a point of inflection at  $x = 4$  because  $f'(x) = g(x)$  changes from decreasing to increasing at  $x = 4$ .

2 :  $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

## 2018 SCORING GUIDELINES

## Question 4

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$  is the rate at which the height of the tree is changing, in meters per year, at time  $t = 6$  years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because  $H$  is differentiable on  $3 \leq t \leq 5$ ,  $H$  is continuous on  $3 \leq t \leq 5$ .

By the Mean Value Theorem, there exists a value  $c$ ,  $3 < c < 5$ , such that  $H'(c) = 2$ .

$$(c) \quad \text{The average height of the tree over the time interval } 2 \leq t \leq 10 \text{ is given by } \frac{1}{10 - 2} \int_2^{10} H(t) \, dt.$$

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) \, dt &\approx \frac{1}{8} \left( \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval  $2 \leq t \leq 10$  is  $\frac{263}{32}$  meters.

$$(d) \quad G(x) = 50 \Rightarrow x = 1$$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is  $\frac{3}{4}$  meter per year.

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$$

$$2 : \begin{cases} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$$

$$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$$

$$3 : \begin{cases} 2 : \frac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$$

Note: max 1/3 [1-0] if no chain rule

## 2018 SCORING GUIDELINES

## Question 5

$$(a) \text{ Area} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} (4^2 - (3 + 2\cos \theta)^2) d\theta$$

$$(b) \frac{dr}{d\theta} = -2\sin \theta \Rightarrow \left. \frac{dr}{d\theta} \right|_{\theta=\pi/2} = -2$$

$$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$$

$$y = r \sin \theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

$$x = r \cos \theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \left. \frac{dy/d\theta}{dx/d\theta} \right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of  $r = 3 + 2\cos \theta$

at  $\theta = \frac{\pi}{2}$  is  $\frac{2}{3}$ .

— OR —

$$y = r \sin \theta = (3 + 2\cos \theta) \sin \theta \Rightarrow \frac{dy}{d\theta} = 3\cos \theta + 2\cos^2 \theta - 2\sin^2 \theta$$

$$x = r \cos \theta = (3 + 2\cos \theta) \cos \theta \Rightarrow \frac{dx}{d\theta} = -3\sin \theta - 4\sin \theta \cos \theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \left. \frac{dy/d\theta}{dx/d\theta} \right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of  $r = 3 + 2\cos \theta$

at  $\theta = \frac{\pi}{2}$  is  $\frac{2}{3}$ .

$$(c) \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = -2\sin \theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin \theta}$$

$$\left. \frac{d\theta}{dt} \right|_{\theta=\pi/3} = 3 \cdot \frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \\ \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \\ 1 : \frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} \\ 1 : \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin \theta} \\ 1 : \text{answer with units} \end{cases}$$

## 2018 SCORING GUIDELINES

## Question 6

(a) The first four nonzero terms are  $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$ .

The general term is  $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$ .

(b) 
$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{n+1} \right| = \left| \frac{x}{3} \right|$$

$\left| \frac{x}{3} \right| < 1$  for  $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for  $f$  is 3.

— OR —

The radius of convergence of the Maclaurin series for  $\ln(1+x)$  is 1, so the series for  $f(x) = x \ln\left(1 + \frac{x}{3}\right)$  converges absolutely for  $\left|\frac{x}{3}\right| < 1$ .

$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for  $f$  is 3.

When  $x = -3$ , the series is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$ , which diverges by comparison to the harmonic series.

When  $x = 3$ , the series is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$ , which converges by the alternating series test.

The interval of convergence of the Maclaurin series for  $f$  is  $-3 < x \leq 3$ .

(c) By the alternating series error bound, an upper bound for  $|P_4(2) - f(2)|$  is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$

2 :  $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

5 :  $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

— OR —

5 :  $\begin{cases} 1 : \text{radius for } \ln(1+x) \text{ series} \\ 1 : \text{substitutes } \frac{x}{3} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

2 :  $\begin{cases} 1 : \text{uses fifth-degree term as error bound} \\ 1 : \text{answer} \end{cases}$

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# **AP<sup>®</sup> Calculus BC**

## **2016 Scoring Guidelines**

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## 2016 SCORING GUIDELINES

## Question 1

|                           |      |      |     |     |     |
|---------------------------|------|------|-----|-----|-----|
| $t$<br>(hours)            | 0    | 1    | 3   | 6   | 8   |
| $R(t)$<br>(liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

Water is pumped into a tank at a rate modeled by  $W(t) = 2000e^{-t^2/20}$  liters per hour for  $0 \leq t \leq 8$ , where  $t$  is measured in hours. Water is removed from the tank at a rate modeled by  $R(t)$  liters per hour, where  $R$  is differentiable and decreasing on  $0 \leq t \leq 8$ . Selected values of  $R(t)$  are shown in the table above. At time  $t = 0$ , there are 50,000 liters of water in the tank.

- (a) Estimate  $R'(2)$ . Show the work that leads to your answer. Indicate units of measure.
- (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- (d) For  $0 \leq t \leq 8$ , is there a time  $t$  when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

$$(a) \quad R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120 \text{ liters/hr}^2$$

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$$

(b) The total amount of water removed is given by  $\int_0^8 R(t) dt$ .

$$\begin{aligned} \int_0^8 R(t) dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since  $R$  is a decreasing function.

$$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$$

$$\begin{aligned} (c) \quad \text{Total} &\approx 50000 + \int_0^8 W(t) dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters} \end{aligned}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$$

(d)  $W(0) - R(0) > 0$ ,  $W(8) - R(8) < 0$ , and  $W(t) - R(t)$  is continuous.

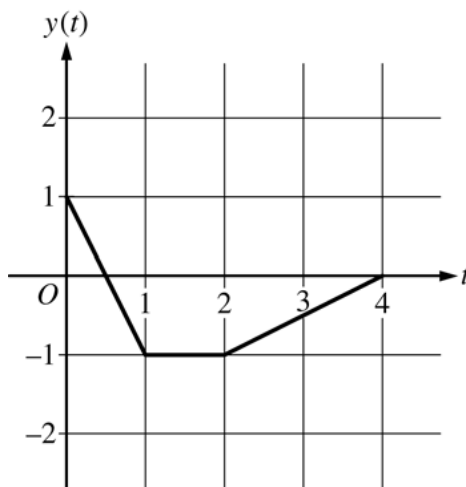
$$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$$

Therefore, the Intermediate Value Theorem guarantees at least one time  $t$ ,  $0 < t < 8$ , for which  $W(t) - R(t) = 0$ , or  $W(t) = R(t)$ .

For this value of  $t$ , the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

## 2016 SCORING GUIDELINES

## Question 2



At time  $t$ , the position of a particle moving in the  $xy$ -plane is given by the parametric functions  $(x(t), y(t))$ , where  $\frac{dx}{dt} = t^2 + \sin(3t^2)$ . The graph of  $y$ , consisting of three line segments, is shown in the figure above. At  $t = 0$ , the particle is at position  $(5, 1)$ .

- Find the position of the particle at  $t = 3$ .
- Find the slope of the line tangent to the path of the particle at  $t = 3$ .
- Find the speed of the particle at  $t = 3$ .
- Find the total distance traveled by the particle from  $t = 0$  to  $t = 2$ .

$$(a) \quad x(3) = x(0) + \int_0^3 x'(t) dt = 5 + 9.377035 = 14.377$$

$$y(3) = -\frac{1}{2}$$

The position of the particle at  $t = 3$  is  $(14.377, -0.5)$ .

$$(b) \quad \text{Slope} = \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$$

$$(c) \quad \text{Speed} = \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969 \text{ (or } 9.968)$$

$$\begin{aligned} (d) \quad \text{Distance} &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2} dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2} dt \\ &= 2.237871 + 2.112003 = 4.350 \text{ (or } 4.349) \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

1 : slope

2 :  $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

3 :  $\begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

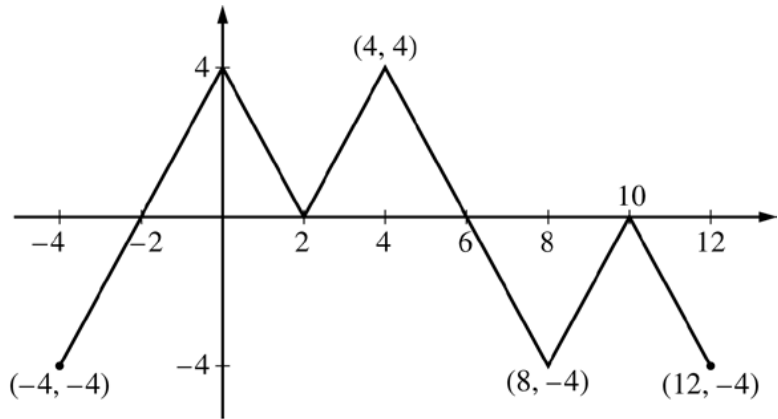
## 2016 SCORING GUIDELINES

## Question 3

The figure above shows the graph of the piecewise-linear function  $f$ . For  $-4 \leq x \leq 12$ , the function  $g$  is defined by

$$g(x) = \int_2^x f(t) \, dt.$$

- (a) Does  $g$  have a relative minimum, a relative maximum, or neither at  $x = 10$ ? Justify your answer.
- (b) Does the graph of  $g$  have a point of inflection at  $x = 4$ ? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of  $g$  on the interval  $-4 \leq x \leq 12$ . Justify your answers.
- (d) For  $-4 \leq x \leq 12$ , find all intervals for which  $g(x) \leq 0$ .

Graph of  $f$ 

- (a) The function  $g$  has neither a relative minimum nor a relative maximum at  $x = 10$  since  $g'(x) = f(x)$  and  $f(x) \leq 0$  for  $8 \leq x \leq 12$ .
- (b) The graph of  $g$  has a point of inflection at  $x = 4$  since  $g'(x) = f(x)$  is increasing for  $2 \leq x \leq 4$  and decreasing for  $4 \leq x \leq 8$ .
- (c)  $g'(x) = f(x)$  changes sign only at  $x = -2$  and  $x = 6$ .

| $x$ | $g(x)$ |
|-----|--------|
| -4  | -4     |
| -2  | -8     |
| 6   | 8      |
| 12  | -4     |

On the interval  $-4 \leq x \leq 12$ , the absolute minimum value is  $g(-2) = -8$  and the absolute maximum value is  $g(6) = 8$ .

- (d)  $g(x) \leq 0$  for  $-4 \leq x \leq 2$  and  $10 \leq x \leq 12$ .

1 :  $g'(x) = f(x)$  in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 :  $\left\{ \begin{array}{l} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \quad \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{array} \right.$

2 : intervals

## 2016 SCORING GUIDELINES

## Question 4

Consider the differential equation  $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ .

(a) Find  $\frac{d^2y}{dx^2}$  in terms of  $x$  and  $y$ .

(b) Let  $y = f(x)$  be the particular solution to the given differential equation whose graph passes through the point  $(-2, 8)$ . Does the graph of  $f$  have a relative minimum, a relative maximum, or neither at the point  $(-2, 8)$ ? Justify your answer.

(c) Let  $y = g(x)$  be the particular solution to the given differential equation with  $g(-1) = 2$ . Find

$$\lim_{x \rightarrow -1} \left( \frac{g(x) - 2}{3(x+1)^2} \right).$$
 Show the work that leads to your answer.

(d) Let  $y = h(x)$  be the particular solution to the given differential equation with  $h(0) = 2$ . Use Euler's method, starting at  $x = 0$  with two steps of equal size, to approximate  $h(1)$ .

$$(a) \quad \frac{d^2y}{dx^2} = 2x - \frac{1}{2} \frac{dy}{dx} = 2x - \frac{1}{2} \left( x^2 - \frac{1}{2}y \right)$$

$$2 : \frac{d^2y}{dx^2} \text{ in terms of } x \text{ and } y$$

$$(b) \quad \left. \frac{dy}{dx} \right|_{(x,y)=(-2,8)} = (-2)^2 - \frac{1}{2} \cdot 8 = 0$$

2 : conclusion with justification

$$\left. \frac{d^2y}{dx^2} \right|_{(x,y)=(-2,8)} = 2(-2) - \frac{1}{2} \left( (-2)^2 - \frac{1}{2} \cdot 8 \right) = -4 < 0$$

Thus, the graph of  $f$  has a relative maximum at the point  $(-2, 8)$ .

$$(c) \quad \lim_{x \rightarrow -1} (g(x) - 2) = 0 \text{ and } \lim_{x \rightarrow -1} 3(x+1)^2 = 0$$

$$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left( \frac{g(x) - 2}{3(x+1)^2} \right) = \lim_{x \rightarrow -1} \left( \frac{g'(x)}{6(x+1)} \right)$$

$$\lim_{x \rightarrow -1} g'(x) = 0 \text{ and } \lim_{x \rightarrow -1} 6(x+1) = 0$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left( \frac{g'(x)}{6(x+1)} \right) = \lim_{x \rightarrow -1} \left( \frac{g''(x)}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$$

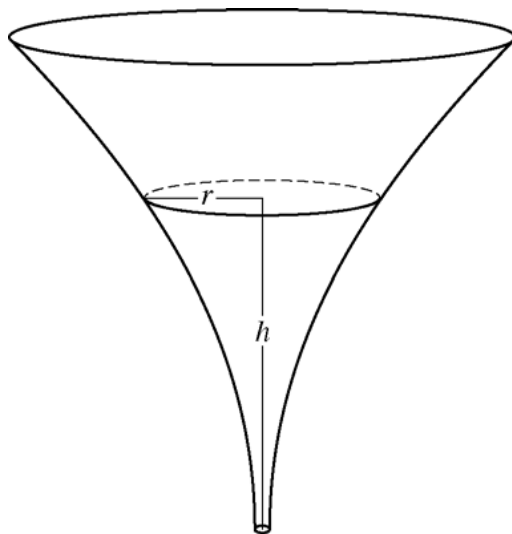
$$(d) \quad h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1) \cdot \frac{1}{2} = \frac{3}{2}$$

$$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{5}{4}$$

$$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$$

## 2016 SCORING GUIDELINES

## Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height  $h$ , the radius of the funnel is given by  $r = \frac{1}{20}(3 + h^2)$ , where  $0 \leq h \leq 10$ . The units of  $r$  and  $h$  are inches.

- (a) Find the average value of the radius of the funnel.
- (b) Find the volume of the funnel.
- (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is  $h = 3$  inches, the radius of the surface of the liquid is decreasing at a rate of  $\frac{1}{5}$  inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\begin{aligned} \text{(a) Average radius} &= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[ 3h + \frac{h^3}{3} \right]_0^{10} \\ &= \frac{1}{200} \left( \left( 30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60} \text{ in} \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_0^{10} \left( \left( \frac{1}{20} \right) (3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh \\ &= \frac{\pi}{400} \left[ 9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} \\ &= \frac{\pi}{400} \left( \left( 90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40} \text{ in}^3 \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(c) } \frac{dr}{dt} &= \frac{1}{20}(2h) \frac{dh}{dt} \\ -\frac{1}{5} &= \frac{3}{10} \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec} \end{aligned}$$

3 :  $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

## 2016 SCORING GUIDELINES

## Question 6

The function  $f$  has a Taylor series about  $x = 1$  that converges to  $f(x)$  for all  $x$  in the interval of convergence.

It is known that  $f(1) = 1$ ,  $f'(1) = -\frac{1}{2}$ , and the  $n$ th derivative of  $f$  at  $x = 1$  is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- (a) Write the first four nonzero terms and the general term of the Taylor series for  $f$  about  $x = 1$ .
- (b) The Taylor series for  $f$  about  $x = 1$  has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- (c) The Taylor series for  $f$  about  $x = 1$  can be used to represent  $f(1.2)$  as an alternating series. Use the first three nonzero terms of the alternating series to approximate  $f(1.2)$ .
- (d) Show that the approximation found in part (c) is within 0.001 of the exact value of  $f(1.2)$ .

$$\begin{aligned} \text{(a)} \quad f(1) = 1, \quad f'(1) = -\frac{1}{2}, \quad f''(1) = \frac{1}{2^2}, \quad f'''(1) = -\frac{2}{2^3} \\ f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots \\ + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots \end{aligned}$$

$$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$$

- (b)  $R = 2$ . The series converges on the interval  $(-1, 3)$ .

When  $x = -1$ , the series is  $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ .

Since the harmonic series diverges, this series diverges.

When  $x = 3$ , the series is  $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$ .

Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is  $-1 < x \leq 3$ .

$$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

$$\text{(c)} \quad f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$$

1 : approximation

- (d) The series for  $f(1.2)$  alternates with terms that decrease in magnitude to 0.

$$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$$

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$