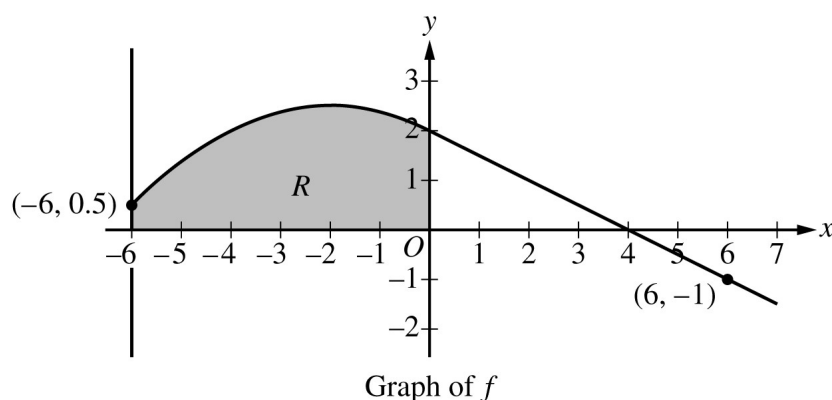


Part B (AB or BC): Graphing calculator not allowed**Question 4****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



The graph of the differentiable function f , shown for $-6 \leq x \leq 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \leq x \leq 7$. Let R be the region in the second quadrant bounded by the graph of f , the vertical line $x = -6$, and the x - and y -axes. Region R has area 12.

Model Solution**Scoring**

- (a) The function g is defined by $g(x) = \int_0^x f(t) dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

$g(-6) = \int_0^{-6} f(t) dt = -\int_{-6}^0 f(t) dt = -12$	$g(-6)$	1 point
$g(4) = \int_0^4 f(t) dt = \frac{1}{2} \cdot 4 \cdot 2 = 4$	$g(4)$	1 point
$g(6) = \int_0^6 f(t) dt = \frac{1}{2} \cdot 4 \cdot 2 - \frac{1}{2} \cdot 2 \cdot 1 = 3$	$g(6)$	1 point

Scoring notes:

- Supporting work is not required for any of these values. However, any supporting work that is shown must be correct to earn the corresponding point.
- Special case: A response that explicitly presents $g(x) = \int_{-6}^x f(t) dt$ does not earn the first point it would have otherwise earned. The response is eligible for all subsequent points for correct answers, or for consistent answers with supporting work.
 - Note: $\int_{-6}^{-6} f(t) dt = 0$, $\int_{-6}^4 f(t) dt = 16$, $\int_{-6}^6 f(t) dt = 15$
- Labeled values may be presented in any order. Unlabeled values are read from left to right and from top to bottom as $g(-6)$, $g(4)$, and $g(6)$, respectively. A response that presents only 1 or 2 values must label them to earn any points.

Total for part (a) 3 points

- (b) For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.

$g'(x) = f(x)$	Fundamental Theorem of Calculus	1 point
$g'(x) = f(x) = 0 \Rightarrow x = 4$	Answer with reason	1 point
Therefore, the graph of g has a critical point at $x = 4$.		

Scoring notes:

- The first point is earned for explicitly making the connection $g' = f$ in this part.
 - A response that writes $g'' = f'$ earns the first point but can only earn the second point by reasoning from $f = 0$.
- A response that does not earn the first point is eligible to earn the second point with an implied application of the FTC (e.g., “Because $g'(4) = 0$, $x = 4$ is a critical point”).
- A response that reports any additional critical points in $0 < x < 6$ does not earn the second point.
 - Any presented critical point outside the interval $0 < x < 6$ will not affect scoring.

Total for part (b) 2 points

- (c) The function h is defined by $h(x) = \int_{-6}^x f'(t) dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

$h(6) = \int_{-6}^6 f'(t) dt = f(6) - f(-6) = -1 - 0.5 = -1.5$	Uses Fundamental Theorem of Calculus	1 point
	$h(6)$ with supporting work	1 point
$h'(x) = f'(x)$, so $h'(6) = f'(6) = -\frac{1}{2}$.	$h'(6)$	1 point
$h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$.	$h''(6)$	1 point

Scoring notes:

- Labeled values may be presented in any order.
- Unlabeled values are read from left to right and from top to bottom as $h(6)$, $h'(6)$, and $h''(6)$, respectively. A response that presents only 1 or 2 values must label them in order to earn any points.
- A response of $h(6) = -1.5$ does not earn either of the first 2 points. A response of $h(6) = f(6) - f(-6)$ earns the first point but not yet the second point.
- A response of $h(6) = -1 - 0.5$ is the minimum work required to earn both of the first 2 points.
- To earn the third point a response must state either $h'(x) = f'(x)$ or $h'(6) = f'(6)$, and provide an answer of $-\frac{1}{2}$.
- The fourth point is earned for a response of $h''(6) = 0$, with or without supporting work.
- A response that has one or more linkage errors does not earn the first point it would have otherwise earned. For example, $h'(x) = f'(6) = -\frac{1}{2}$ does not earn the third point but is eligible for the fourth point even in the presence of another linkage error, such as $h''(x) = f''(6) = 0$.

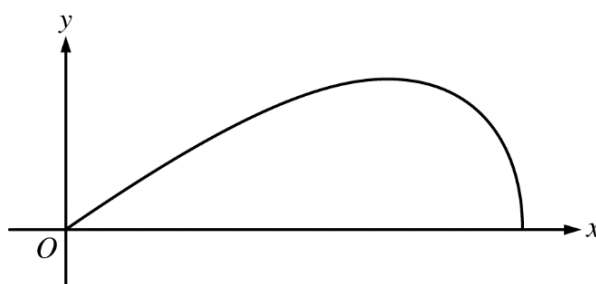
Total for part (c) 4 points

Total for question 4 9 points

Part B (AB or BC): Graphing calculator not allowed**Question 3****9 points****General Scoring Notes**

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant.

The figure above shows the region in the first quadrant bounded by the x -axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c . Each spinning toy is in the shape of the solid generated when such a region is revolved about the x -axis. Both x and y are measured in inches.

Model Solution	Scoring
(a) Find the area of the region in the first quadrant bounded by the x -axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.	
$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0, x = 2$ $\text{Area} = \int_0^2 6x\sqrt{4 - x^2} \, dx$	Integrand 1 point
Let $u = 4 - x^2$. $du = -2x \, dx \Rightarrow -\frac{1}{2} du = x \, dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2} \, dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u} \, du = -3\int_4^0 u^{1/2} \, du = 3\int_0^4 u^{1/2} \, du$ $= 2u^{3/2} \Big _{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative 1 point
The area of the region is 16 square inches.	Answer 1 point

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant A . If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u -substitution and have incorrect limits of integration or do not change the limits of integration from x - to u -values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

(b)

It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$.

Sets $\frac{dy}{dx} = 0$ **1 point**

$$\frac{dy}{dx} = \frac{c(4 - 2x^2)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$$

$$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - (\sqrt{2})^2} = 2c$$

$$2c = 1.2 \Rightarrow c = 0.6$$

Answer **1 point**

Scoring notes:

- The first point is earned for setting $\frac{dy}{dx} = 0$, $\frac{c(4 - 2x^2)}{\sqrt{4 - x^2}} = 0$, or $c(4 - 2x^2) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

- (c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$\text{Volume} = \int_0^2 \pi \left(cx\sqrt{4-x^2} \right)^2 dx = \pi c^2 \int_0^2 x^2 (4-x^2) dx$	Form of the integrand	1 point
	Limits and constant	1 point
$= \pi c^2 \int_0^2 (4x^2 - x^4) dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \right) \Big _0^2$	Antiderivative	1 point
$= \pi c^2 \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	1 point

Scoring notes:

- The first point is earned for presenting an integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A . Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π , the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2 (cx\sqrt{4-x^2})^2 dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A(x\sqrt{4-x^2})^2$ for any nonzero A . If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points

2017**AP[®]** **CollegeBoard**

AP Calculus BC

Scoring Guidelines

2017 SCORING GUIDELINES

Question 1

(a) Volume = $\int_0^{10} A(h) \, dh$ $\approx (2 - 0) \cdot A(0) + (5 - 2) \cdot A(2) + (10 - 5) \cdot A(5)$ $= 2 \cdot 50.3 + 3 \cdot 14.4 + 5 \cdot 6.5$ $= 176.3$ cubic feet (b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing. (c) $\int_0^{10} f(h) \, dh = 101.325338$ The volume is 101.325 cubic feet. (d) Using the model, $V(h) = \int_0^h f(x) \, dx$. $\begin{aligned} \left. \frac{dV}{dt} \right _{h=5} &= \left[\frac{dV}{dh} \cdot \frac{dh}{dt} \right]_{h=5} \\ &= \left[f(h) \cdot \frac{dh}{dt} \right]_{h=5} \\ &= f(5) \cdot 0.26 = 1.694419 \end{aligned}$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	1 : units in parts (a), (c), and (d) 2 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$ 1 : overestimate with reason 2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ 3 : $\begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$
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2017 SCORING GUIDELINES

Question 2

$$(a) \frac{1}{2} \int_0^{\pi/2} (f(\theta))^2 d\theta = 0.648414$$

The area of R is 0.648.

$$(b) \int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \frac{1}{2} \int_0^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$$

— OR —

$$\int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \int_k^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$$

$$(c) w(\theta) = g(\theta) - f(\theta)$$

$$w_A = \frac{\int_0^{\pi/2} w(\theta) d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$$(d) w(\theta) = w_A \text{ for } 0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \theta = 0.517688$$

$$w(\theta) = w_A \text{ at } \theta = 0.518 \text{ (or 0.517).}$$

$$w'(0.518) < 0 \Rightarrow w(\theta) \text{ is decreasing at } \theta = 0.518.$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{integral expression} \\ \quad \text{for one region} \\ 1 : \text{equation} \end{cases}$$

$$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$$

$$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$$

2017 SCORING GUIDELINES

Question 3

$$(a) f(-6) = f(-2) + \int_{-2}^{-6} f'(x) dx = 7 - \int_{-6}^{-2} f'(x) dx = 7 - 4 = 3$$

$$f(5) = f(-2) + \int_{-2}^5 f'(x) dx = 7 - 2\pi + 3 = 10 - 2\pi$$

$$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$$

$$(b) f'(x) > 0 \text{ on the intervals } [-6, -2) \text{ and } (2, 5).$$

Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

$$2 : \text{answer with justification}$$

$$(c) \text{ The absolute minimum will occur at a critical point where } f'(x) = 0 \text{ or at an endpoint.}$$

$$f'(x) = 0 \Rightarrow x = -2, x = 2$$

$$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$$(d) f''(-5) = \frac{2 - 0}{-6 - (-2)} = -\frac{1}{2}$$

$$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \text{with explanation} \end{cases}$$

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2 \text{ and } \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1$$

$f''(3)$ does not exist because

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3}.$$

2017 SCORING GUIDELINES

Question 4

$$(a) \quad H'(0) = -\frac{1}{4}(91 - 27) = -16$$

$$H(0) = 91$$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$(b) \quad \frac{d^2H}{dt^2} = -\frac{1}{4} \frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$$

$$H > 27 \text{ for } t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0 \text{ for } t > 0$$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

$$(c) \quad \frac{dG}{(G - 27)^{2/3}} = -dt$$

$$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1) dt$$

$$3(G - 27)^{1/3} = -t + C$$

$$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$$

$$3(G - 27)^{1/3} = 12 - t$$

$$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3 \text{ for } 0 \leq t < 10$$

The internal temperature of the potato at time $t = 3$ minutes is

$$27 + \left(\frac{12 - 3}{3}\right)^3 = 54 \text{ degrees Celsius.}$$

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

1 : underestimate with reason

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

2017 SCORING GUIDELINES

Question 5

$$(a) f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2}$$

$$f'(3) = \frac{(-3)(5)}{(18 - 21 + 5)^2} = -\frac{15}{4}$$

$$(b) f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2} = 0 \Rightarrow x = \frac{7}{4}$$

The only critical point in the interval $1 < x < 2.5$ has x -coordinate $\frac{7}{4}$.

f' changes sign from positive to negative at $x = \frac{7}{4}$.

Therefore, f has a relative maximum at $x = \frac{7}{4}$.

$$\begin{aligned} (c) \int_5^\infty f(x) dx &= \lim_{b \rightarrow \infty} \int_5^b \frac{3}{2x^2 - 7x + 5} dx = \lim_{b \rightarrow \infty} \int_5^b \left(\frac{2}{2x - 5} - \frac{1}{x - 1} \right) dx \\ &= \lim_{b \rightarrow \infty} \left[\ln(2x - 5) - \ln(x - 1) \right]_5^b = \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2x - 5}{x - 1} \right) \right]_5^b \\ &= \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2b - 5}{b - 1} \right) - \ln \left(\frac{5}{4} \right) \right] = \ln 2 - \ln \left(\frac{5}{4} \right) = \ln \left(\frac{8}{5} \right) \end{aligned}$$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\int_5^\infty \frac{3}{2x^2 - 7x + 5} dx$ converges.

— OR —

$$\frac{3}{2n^2 - 7n + 5} > 0 \text{ and } \frac{1}{n^2} > 0 \text{ for } n \geq 5.$$

Since $\lim_{n \rightarrow \infty} \frac{\frac{3}{2n^2 - 7n + 5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\sum_{n=5}^\infty \frac{1}{n^2}$ converges,

the series $\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$ converges by the limit comparison test.

2 : $f'(3)$

2 : $\begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \text{with justification} \end{cases}$

3 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

2 : answer with conditions

2017 SCORING GUIDELINES

Question 6

(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

$$\begin{aligned} \text{(c)} \quad \int_0^x f(t) dt &= \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt \\ &= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x} \\ &= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots \end{aligned}$$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded

by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

$$\text{Thus, } \left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}.$$

$$3 : \begin{cases} 1 : f''(0), f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$$

2 : converges conditionally
with reason

$$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$$

1 : error bound

AP[®] Calculus BC

2015 Scoring Guidelines

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2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- (c) At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t) dt = 76.570$

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$
 Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

2 : $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t , $0 \leq t \leq 8$, is $30 + \int_0^t [R(x) - D(x)] dx$.

3 : $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

(d) $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

2015 SCORING GUIDELINES

Question 2

At time $t \geq 0$, a particle moving along a curve in the xy -plane has position $(x(t), y(t))$ with velocity vector $v(t) = (\cos(t^2), e^{0.5t})$. At $t = 1$, the particle is at the point $(3, 5)$.

- (a) Find the x -coordinate of the position of the particle at time $t = 2$.
- (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

(a) $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$ (or 2.556)

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

$$2 : \begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$$

(c) Speed = $\sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

$$2 : \begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$$

(d) Distance = $\int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$ (or 1.594)

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

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Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	-220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by

$$B(t) = t^3 - 6t^2 + 300, \text{ where } t \text{ is measured in minutes and } B(t) \text{ is measured in meters per minute.}$$

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

$$(a) \quad v'(16) \approx \frac{240 - 200}{20 - 12} = 5 \text{ meters/min}^2$$

1 : approximation

(b) $\int_0^{40} |v(t)| dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \leq t \leq 40$ minutes.

3 : $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

(c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.

$$B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$$

2 : $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

$$(d) \quad \text{Avg vel} = \frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$$

$$\begin{aligned} &= \frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10} \\ &= \frac{1}{10} \left[\frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min} \end{aligned}$$

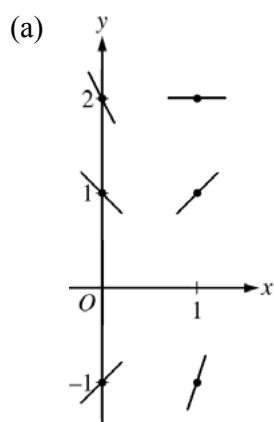
3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

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Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.



$$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$$

(c) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$$2 : \begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

$$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$$

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Question 5

Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x -coordinate is 4.
- (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.
- (c) Find the value of k for which f has a critical point at $x = -5$.
- (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f .

Find $\int f(x) dx$.

$$(a) \quad f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4} \quad f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$$

An equation for the line tangent to the graph of f at the point whose x -coordinate is 4 is $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$.

$$(b) \quad f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2} \quad f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

$$(c) \quad f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$$

$$(d) \quad \frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

1 : answer

4 : $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

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Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \dots + \frac{(-3)^{n-1}}{n} x^n + \dots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find R .
- (b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.
- (c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

- (a) Let a_n be the n th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

- (b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for e^x are

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\begin{aligned} & \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(x - \frac{3}{2}x^2 + 3x^3 - \dots \right) \\ &= x - \frac{1}{2}x^2 + 2x^3 + \dots \end{aligned}$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$

about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

$$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$$

$$3 : \begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$$

$$3 : \begin{cases} 1 : \text{first four nonzero terms} \\ \quad \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$$