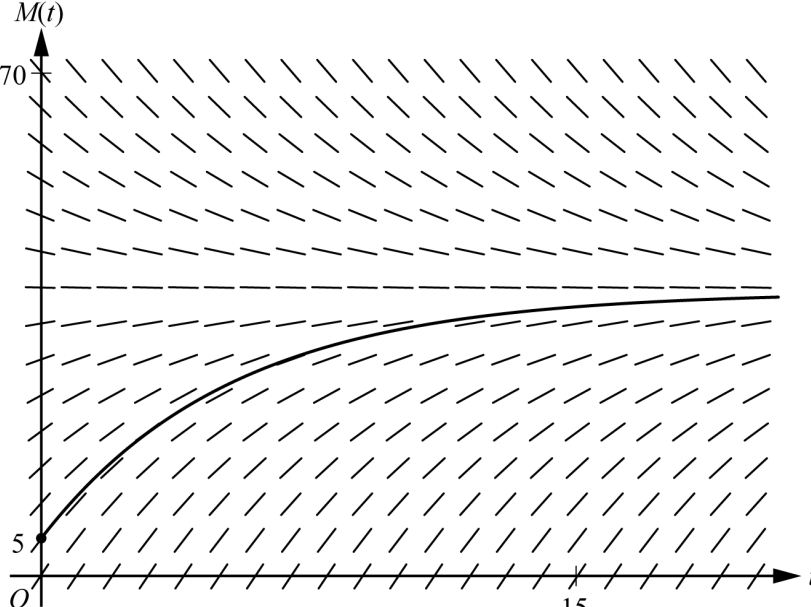


Part B (AB or BC): Graphing calculator not allowed**Question 3****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

Model Solution	Scoring
(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$. 	Solution curve 1 point

Scoring notes:

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

Total Score (a) 1 point

- (b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

$\left. \frac{dM}{dt} \right _{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$	$\left. \frac{dM}{dt} \right _{t=0}$ 1 point
The tangent line equation is $y = 5 + \frac{35}{4}(t - 0)$. $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$ The temperature of the milk at time $t = 2$ minutes is approximately 22.5° Celsius.	Approximation 1 point

Scoring notes:

- The value of the slope may appear in a tangent line equation or approximation.
- A response of $5 + \frac{35}{4} \cdot 2$ is the minimal response to earn both points.
- A response of $\frac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.
- In order to earn the second point the response must present an approximation found by using a tangent line that:
 - passes through the point $(0, 5)$ and
 - has slope $\frac{35}{4}$ or a nonzero slope that is declared to be the value of $\frac{dM}{dt}$.
- An unsupported approximation does not earn the second point.
- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

Total for part (b) 2 points

- (c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M . Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

$\frac{d^2M}{dt^2} = -\frac{1}{4} \frac{dM}{dt} = -\frac{1}{4} \left(\frac{1}{4}(40 - M) \right) = -\frac{1}{16}(40 - M)$	$\frac{d^2M}{dt^2}$ 1 point
Because $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.	Overestimate with reason 1 point

Scoring notes:

- The first point is earned for either $\frac{d^2M}{dt^2} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$ or $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.
- A response that presents an expression for $\frac{d^2M}{dt^2}$ in terms of $\frac{dM}{dt}$ but fails to continue to an expression in terms of M (i.e., $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt}$) does not earn the first point but is eligible for the second point.
- If the response presents an expression for $\frac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.
 - Special case: A response that presents $\frac{d^2M}{dt^2} = \frac{1}{16}(40 - M)$ does not earn the first point but is eligible to earn the second point for a consistent answer and reason.
- To earn the second point a response must include $\frac{d^2M}{dt^2} < 0$, or $\frac{dM}{dt}$ is decreasing, or the graph of M is concave down, as well as the conclusion that the approximation is an overestimate.
- A response that presents an argument based on $\frac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

Total for part (c) 2 points

- (d)** Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

$\frac{dM}{40 - M} = \frac{1}{4} dt$ $\int \frac{dM}{40 - M} = \int \frac{1}{4} dt$	Separates variables	1 point
$-\ln 40 - M = \frac{1}{4}t + C$	Finds antiderivatives	1 point
$-\ln 40 - 5 = 0 + C \Rightarrow C = -\ln 35$ $M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow 40 - M = 40 - M$ $-\ln(40 - M) = \frac{1}{4}t - \ln 35$ $\ln(40 - M) = -\frac{1}{4}t + \ln 35$	Constant of integration and uses initial condition	1 point

$$40 - M = 35e^{-t/4}$$

$$M = 40 - 35e^{-t/4}$$

Solves for M **1 point****Scoring notes:**

- A response with no separation of variables earns 0 out of 4 points.
- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is eligible for all 4 points.
- A response with no constant of integration can earn at most the first 2 points.
- A response is eligible for the third point only if it has earned the first 2 points.
 - Special Case: A response that presents $+\ln(40 - M) = \frac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.
- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting 0 for t and 5 for M .
- A response is eligible for the fourth point only if it has earned the first 3 points.
- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

Total for part (d) 4 points**Total for question 3 9 points****2017****AP[®]****CollegeBoard**

AP Calculus BC

Scoring Guidelines

2017 SCORING GUIDELINES

Question 1

(a) Volume = $\int_0^{10} A(h) \, dh$ $\approx (2 - 0) \cdot A(0) + (5 - 2) \cdot A(2) + (10 - 5) \cdot A(5)$ $= 2 \cdot 50.3 + 3 \cdot 14.4 + 5 \cdot 6.5$ $= 176.3$ cubic feet (b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing. (c) $\int_0^{10} f(h) \, dh = 101.325338$ The volume is 101.325 cubic feet. (d) Using the model, $V(h) = \int_0^h f(x) \, dx$. $\begin{aligned} \left. \frac{dV}{dt} \right _{h=5} &= \left[\frac{dV}{dh} \cdot \frac{dh}{dt} \right]_{h=5} \\ &= \left[f(h) \cdot \frac{dh}{dt} \right]_{h=5} \\ &= f(5) \cdot 0.26 = 1.694419 \end{aligned}$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	1 : units in parts (a), (c), and (d) 2 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$ 1 : overestimate with reason 2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ 3 : $\begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$
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2017 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2} \int_0^{\pi/2} (f(\theta))^2 d\theta = 0.648414$

The area of R is 0.648.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \frac{1}{2} \int_0^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$

— OR —

$$\int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \int_k^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$$

2 : $\begin{cases} 1 : \text{integral expression} \\ \quad \text{for one region} \\ 1 : \text{equation} \end{cases}$

(c) $w(\theta) = g(\theta) - f(\theta)$

$$w_A = \frac{\int_0^{\pi/2} w(\theta) d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

3 : $\begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$

(d) $w(\theta) = w_A$ for $0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

2 : $\begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

2017 SCORING GUIDELINES

Question 3

$$(a) \quad f(-6) = f(-2) + \int_{-2}^{-6} f'(x) \, dx = 7 - \int_{-6}^{-2} f'(x) \, dx = 7 - 4 = 3$$

$$f(5) = f(-2) + \int_{-2}^5 f'(x) \, dx = 7 - 2\pi + 3 = 10 - 2\pi$$

$$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$$

$$(b) \quad f'(x) > 0 \text{ on the intervals } [-6, -2) \text{ and } (2, 5).$$

Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

$$(c) \quad \text{The absolute minimum will occur at a critical point where } f'(x) = 0 \text{ or at an endpoint.}$$

$$f'(x) = 0 \Rightarrow x = -2, x = 2$$

$$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$$(d) \quad f''(-5) = \frac{2 - 0}{-6 - (-2)} = -\frac{1}{2}$$

$$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \text{with explanation} \end{cases}$$

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2 \text{ and } \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1$$

$f''(3)$ does not exist because

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3}.$$

2017 SCORING GUIDELINES

Question 4

$$(a) \quad H'(0) = -\frac{1}{4}(91 - 27) = -16$$

$$H(0) = 91$$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$(b) \quad \frac{d^2H}{dt^2} = -\frac{1}{4} \frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$$

$$H > 27 \text{ for } t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0 \text{ for } t > 0$$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

$$(c) \quad \frac{dG}{(G - 27)^{2/3}} = -dt$$

$$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1) dt$$

$$3(G - 27)^{1/3} = -t + C$$

$$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$$

$$3(G - 27)^{1/3} = 12 - t$$

$$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3 \text{ for } 0 \leq t < 10$$

The internal temperature of the potato at time $t = 3$ minutes is

$$27 + \left(\frac{12 - 3}{3}\right)^3 = 54 \text{ degrees Celsius.}$$

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

1 : underestimate with reason

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

2017 SCORING GUIDELINES

Question 5

$$(a) f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2}$$

$$f'(3) = \frac{(-3)(5)}{(18 - 21 + 5)^2} = -\frac{15}{4}$$

$$(b) f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2} = 0 \Rightarrow x = \frac{7}{4}$$

The only critical point in the interval $1 < x < 2.5$ has x -coordinate $\frac{7}{4}$.

f' changes sign from positive to negative at $x = \frac{7}{4}$.

Therefore, f has a relative maximum at $x = \frac{7}{4}$.

$$\begin{aligned} (c) \int_5^\infty f(x) dx &= \lim_{b \rightarrow \infty} \int_5^b \frac{3}{2x^2 - 7x + 5} dx = \lim_{b \rightarrow \infty} \int_5^b \left(\frac{2}{2x - 5} - \frac{1}{x - 1} \right) dx \\ &= \lim_{b \rightarrow \infty} \left[\ln(2x - 5) - \ln(x - 1) \right]_5^b = \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2x - 5}{x - 1} \right) \right]_5^b \\ &= \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2b - 5}{b - 1} \right) - \ln \left(\frac{5}{4} \right) \right] = \ln 2 - \ln \left(\frac{5}{4} \right) = \ln \left(\frac{8}{5} \right) \end{aligned}$$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\int_5^\infty \frac{3}{2x^2 - 7x + 5} dx$ converges.

— OR —

$$\frac{3}{2n^2 - 7n + 5} > 0 \text{ and } \frac{1}{n^2} > 0 \text{ for } n \geq 5.$$

Since $\lim_{n \rightarrow \infty} \frac{\frac{3}{2n^2 - 7n + 5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\sum_{n=5}^\infty \frac{1}{n^2}$ converges,

the series $\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$ converges by the limit comparison test.

2 : $f'(3)$

2 : $\begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \text{with justification} \end{cases}$

3 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

2 : answer with conditions

2017 SCORING GUIDELINES

Question 6

(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

$$\begin{aligned} \text{(c)} \quad \int_0^x f(t) dt &= \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt \\ &= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x} \\ &= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots \end{aligned}$$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded

by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

$$\text{Thus, } \left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}.$$

$$3 : \begin{cases} 1 : f''(0), f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$$

2 : converges conditionally
with reason

$$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$$

1 : error bound

AP[®] Calculus BC

2015 Scoring Guidelines

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2015 SCORING GUIDELINES

Question 1

The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

- (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- (c) At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

(a) $\int_0^8 R(t) dt = 76.570$

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $R(3) - D(3) = -0.313632 < 0$
 Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at time $t = 3$ hours.

2 : $\begin{cases} 1 : \text{considers } R(3) \text{ and } D(3) \\ 1 : \text{answer and reason} \end{cases}$

(c) The amount of water in the pipe at time t , $0 \leq t \leq 8$, is $30 + \int_0^t [R(x) - D(x)] dx$.

3 : $\begin{cases} 1 : \text{considers } R(t) - D(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

$R(t) - D(t) = 0 \Rightarrow t = 0, 3.271658$

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours.

(d) $30 + \int_0^w [R(t) - D(t)] dt = 50$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

2015 SCORING GUIDELINES

Question 2

At time $t \geq 0$, a particle moving along a curve in the xy -plane has position $(x(t), y(t))$ with velocity vector $v(t) = (\cos(t^2), e^{0.5t})$. At $t = 1$, the particle is at the point $(3, 5)$.

- (a) Find the x -coordinate of the position of the particle at time $t = 2$.
- (b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?
- (c) Find the time at which the speed of the particle is 3.
- (d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

(a) $x(2) = 3 + \int_1^2 \cos(t^2) dt = 2.557$ (or 2.556)

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{0.5t}}{\cos(t^2)}$

$$\frac{e^{0.5t}}{\cos(t^2)} = 2$$

$$t = 0.840$$

$$2 : \begin{cases} 1 : \text{slope in terms of } t \\ 1 : \text{answer} \end{cases}$$

(c) Speed $= \sqrt{\cos^2(t^2) + e^t}$

$$\sqrt{\cos^2(t^2) + e^t} = 3$$

$$t = 2.196$$
 (or 2.195)

$$2 : \begin{cases} 1 : \text{speed in terms of } t \\ 1 : \text{answer} \end{cases}$$

(d) Distance $= \int_0^1 \sqrt{\cos^2(t^2) + e^t} dt = 1.595$ (or 1.594)

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

2015 SCORING GUIDELINES

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	-220	150

Johanna jogs along a straight path. For $0 \leq t \leq 40$, Johanna's velocity is given by a differentiable function v . Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

- (a) Use the data in the table to estimate the value of $v'(16)$.
- (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)| dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)| dt$ using a right Riemann sum with the four subintervals indicated in the table.

- (c) Bob is riding his bicycle along the same path. For $0 \leq t \leq 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

- (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \leq t \leq 10$.

(a) $v'(16) \approx \frac{240 - 200}{20 - 12} = 5 \text{ meters/min}^2$

1 : approximation

- (b) $\int_0^{40} |v(t)| dt$ is the total distance Johanna jogs, in meters, over the time interval $0 \leq t \leq 40$ minutes.

3 : $\begin{cases} 1 : \text{explanation} \\ 1 : \text{right Riemann sum} \\ 1 : \text{approximation} \end{cases}$

$$\begin{aligned} \int_0^{40} |v(t)| dt &\approx 12 \cdot |v(12)| + 8 \cdot |v(20)| + 4 \cdot |v(24)| + 16 \cdot |v(40)| \\ &= 12 \cdot 200 + 8 \cdot 240 + 4 \cdot 220 + 16 \cdot 150 \\ &= 2400 + 1920 + 880 + 2400 \\ &= 7600 \text{ meters} \end{aligned}$$

- (c) Bob's acceleration is $B'(t) = 3t^2 - 12t$.
 $B'(5) = 3(25) - 12(5) = 15 \text{ meters/min}^2$

2 : $\begin{cases} 1 : \text{uses } B'(t) \\ 1 : \text{answer} \end{cases}$

(d) Avg vel $= \frac{1}{10} \int_0^{10} (t^3 - 6t^2 + 300) dt$

$$\begin{aligned} &= \frac{1}{10} \left[\frac{t^4}{4} - 2t^3 + 300t \right]_0^{10} \\ &= \frac{1}{10} \left[\frac{10000}{4} - 2000 + 3000 \right] = 350 \text{ meters/min} \end{aligned}$$

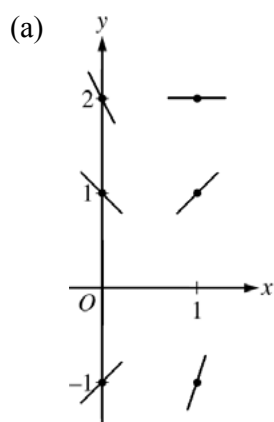
3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

2015 SCORING GUIDELINES

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.



$$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$$

(c) $\left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$$2 : \begin{cases} 1 : \text{considers } \left. \frac{dy}{dx} \right|_{(x,y)=(2,3)} \\ 1 : \text{conclusion with justification} \end{cases}$$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$$2x - y = m$$

$$2x - (mx + b) = m$$

$$(2 - m)x - (m + b) = 0$$

$$2 - m = 0 \Rightarrow m = 2$$

$$b = -m \Rightarrow b = -2$$

$$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$$

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Question 5

Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x -coordinate is 4.
- (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.
- (c) Find the value of k for which f has a critical point at $x = -5$.
- (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f .

Find $\int f(x) dx$.

$$(a) \quad f(4) = \frac{1}{4^2 - 3 \cdot 4} = \frac{1}{4} \quad f'(4) = \frac{3 - 2 \cdot 4}{(4^2 - 3 \cdot 4)^2} = -\frac{5}{16}$$

An equation for the line tangent to the graph of f at the point whose x -coordinate is 4 is $y = -\frac{5}{16}(x - 4) + \frac{1}{4}$.

$$(b) \quad f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2} \quad f'(2) = \frac{4 - 2 \cdot 2}{(2^2 - 4 \cdot 2)^2} = 0$$

$f'(x)$ changes sign from positive to negative at $x = 2$.
Therefore, f has a relative maximum at $x = 2$.

$$(c) \quad f'(-5) = \frac{k - 2 \cdot (-5)}{((-5)^2 - k \cdot (-5))^2} = 0 \Rightarrow k = -10$$

$$(d) \quad \frac{1}{x^2 - 6x} = \frac{1}{x(x - 6)} = \frac{A}{x} + \frac{B}{x - 6} \Rightarrow 1 = A(x - 6) + Bx$$

$$x = 0 \Rightarrow 1 = A \cdot (-6) \Rightarrow A = -\frac{1}{6}$$

$$x = 6 \Rightarrow 1 = B \cdot (6) \Rightarrow B = \frac{1}{6}$$

$$\frac{1}{x(x - 6)} = \frac{-1/6}{x} + \frac{1/6}{x - 6}$$

$$\int f(x) dx = \int \left(\frac{-1/6}{x} + \frac{1/6}{x - 6} \right) dx$$

$$= -\frac{1}{6} \ln|x| + \frac{1}{6} \ln|x - 6| + C = \frac{1}{6} \ln \left| \frac{x - 6}{x} \right| + C$$

2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

2 : $\begin{cases} 1 : \text{considers } f'(2) \\ 1 : \text{answer with justification} \end{cases}$

1 : answer

4 : $\begin{cases} 2 : \text{partial fraction decomposition} \\ 2 : \text{general antiderivative} \end{cases}$

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Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

- (a) Use the ratio test to find R .
- (b) Write the first four nonzero terms of the Maclaurin series for f' , the derivative of f . Express f' as a rational function for $|x| < R$.
- (c) Write the first four nonzero terms of the Maclaurin series for e^x . Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

- (a) Let a_n be the n th term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n \rightarrow \infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

- (b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

- (c) The first four nonzero terms of the Maclaurin series for e^x are

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\begin{aligned} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \left(x - \frac{3}{2}x^2 + 3x^3 - \cdots \right) \\ = x - \frac{1}{2}x^2 + 2x^3 + \cdots \end{aligned}$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$

about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

$$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$$

$$3 : \begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$$

$$3 : \begin{cases} 1 : \text{first four nonzero terms} \\ \quad \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$$