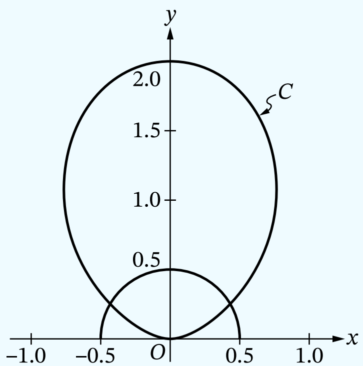
Part A (BC): Graphing calculator required

Question 2 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

	Model Solution	Scoring	
A	Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.		
	$\left.\frac{dr}{d\theta}\right\|_{\theta=1.3} = 1.031003$ The rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$ is 1.031.	Answer with setup	**Point 1 (P1)**

Scoring Notes for Part A

- An exact answer of $\left.\frac{dr}{d\theta}\right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3)$ earns **P1**.
- To earn **P1**, a response must indicate differentiation of r and provide the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point.
  - Examples of responses with correct communication include $\left.\frac{dr}{d\theta}\right|_{\theta=1.3} = 1.031$ and $r'(1.3) = 1.031$.
  - Responses with incorrect communication, such as $r'(\theta) = 1.031$, $r' = 1.031$, or $\frac{dr}{d\theta} = 1.031$, are sufficient to earn **P1**.

	Model Solution	Scoring	
B	Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.		
	For $0 \le \theta \le \pi$, $r(\theta) = \frac{1}{2}$ for $\theta = \theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta = \theta_2 = \frac{5\pi}{6} = 2.617994$.	Integrand including $(r(\theta))^2$	**Point 2 (P2)**
	$\frac{1}{2}\int_{\theta_1}^{\theta_2}\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right)d\theta$	Integrand	**Point 3 (P3)**
	$= 2.066769$ The area is 2.067 (or 2.066).	Answer	**Point 4 (P4)**

Scoring Notes for Part B

- **P2** is earned for a definite integral including $(r(\theta))^2$, such as $\int_a^b\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right)d\theta$ or $\int_a^b (r(\theta))^2\,d\theta$, with or without the differential $d\theta$.
- **P3** is earned for a definite integral (or integrals) with a correct integrand, such as $\int_a^b\left((r(\theta))^2 - \left(\frac{1}{2}\right)^2\right)d\theta$ or $\int_a^b (r(\theta))^2\,d\theta - \int_c^d \left(\frac{1}{2}\right)^2 d\theta$, with or without the differential $d\theta$.
- The limits $\theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta_2 = \frac{5\pi}{6} = 2.617994$ and the factor $\frac{1}{2}$ are assessed in **P4**, not in **P2** or **P3**.
- **P4** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

- Incorrect or unclear communication between the correct integral and the correct answer is treated as scratch work and is not considered in scoring. For example:
  - $\int_{\pi/6}^{5\pi/6}\left((r(\theta))^2-\left(\frac{1}{2}\right)^2\right)d\theta = 4.133538$ so the area is 2.067.

    Note: This response earns **P2** and **P3** for the integral. It also earns **P4** for the correct answer.
  - $\int_{\pi/6}^{5\pi/6}\left((r(\theta))^2-\left(\frac{1}{2}\right)^2\right)d\theta = 2.067$

    Note: This response earns **P2** and **P3** for the integral. It also earns **P4** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)
- **Special case:** An indefinite integral with a correct integrand does not earn **P2**, earns **P3**, and is eligible to earn **P4** with a correct answer.
- A response of $\int_{\pi/6}^{\pi/2}\left((r(\theta))^2-\left(\frac{1}{2}\right)^2\right)d\theta = 2.067$, using the symmetry of the region, earns **P2**, **P3**, and **P4**.

C It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

For $0 \le \theta \le \frac{\pi}{2}$, the curve C is in the first quadrant. Thus, a point on the curve will be farthest away from the y-axis when the x-coordinate attains its maximum value. This will either occur when $\frac{dx}{d\theta} = 0$ or at an endpoint of the interval $0 \le \theta \le \frac{\pi}{2}$. $\frac{dx}{d\theta} = 0$	Considers $\frac{dx}{d\theta} = 0$	**Point 5 (P5)**
$\Rightarrow \theta = 0.955317$ θ | $x(\theta) = r(\theta)\cos\theta$ 0 | 0 0.955317 | 0.769800 $\frac{\pi}{2}$ | 0	Justification	**Point 6 (P6)**
Therefore, the value of θ for which the point on the curve is farthest from the y-axis is 0.955.	Answer with supporting work	**Point 7 (P7)**

Scoring Notes for Part C

- **P5** is earned for considering $\frac{dx}{d\theta} = 0$. **P5** is not earned by just presenting $\theta = 0.955317$.

  A response that discusses the sign of $\frac{dx}{d\theta}$ changing or uses the phrase "critical points of $x(\theta)$" also earns **P5**.
- The value $\theta = 0.955317$ might be presented as $\arccos\left(\frac{1}{\sqrt{3}}\right)$, $\arcsin\left(\sqrt{\frac{2}{3}}\right)$, or $\arctan\left(\sqrt{2}\right)$.
- To earn **P6** using a candidates test, a response must make a global argument by correctly evaluating $x(\theta)$ at $\theta = 0$, $\theta = 0.955317$, and $\theta = \frac{\pi}{2}$. The evaluations must be correct to the first digit after the decimal, rounded or truncated.
- Alternate justifications:
  - $\frac{dx}{d\theta} > 0$ for $0 < \theta < 0.955$, and $\frac{dx}{d\theta} < 0$ for $0.955 < \theta < \frac{\pi}{2}$. Therefore, $\theta = 0.955$ is the location of the absolute maximum for $x(\theta)$ on the interval $0 \le \theta \le \frac{\pi}{2}$.
  - Because $\frac{dx}{d\theta}$ changes sign from positive to negative at $\theta = 0.955$ (this might be presented as "$\frac{dx}{d\theta} > 0$ for $\theta < 0.955$, and $\frac{dx}{d\theta} < 0$ for $\theta > 0.955$"), it is the location of a relative maximum for $x(\theta)$. And because $\theta = 0.955$ is the only critical point of $x(\theta)$ in the interval $0 \le \theta \le \frac{\pi}{2}$, it is the location of the absolute maximum for $x(\theta)$ on the interval.
  - Because $\left.\frac{dx}{d\theta}\right|_{\theta=0.955} = 0$ and $\left.\frac{d^2x}{d\theta^2}\right|_{\theta=0.955} < 0$, $\theta = 0.955$ is the location of a relative maximum for $x(\theta)$. And because $\theta = 0.955$ is the only critical point of $x(\theta)$ in the interval $0 \le \theta \le \frac{\pi}{2}$, it is the location of the absolute maximum for $x(\theta)$ on the interval.
- A response that presents only a local argument (such as a First Derivative Test or a Second Derivative Test) or an incorrect global argument does not earn **P6** but is eligible for **P7** with the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

D	A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.	
	$\left.\frac{dr}{dt}\right\|_{\theta=1.3} = \left.\left(\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}\right)\right\|_{\theta=1.3} = 1.031003\cdot 15 = 15.465041$	$\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$ **Point 8 (P8)**
	The particle's distance from the origin changes at a rate of 15.465.	Answer **Point 9 (P9)**

Scoring Notes for Part D

- To earn **P8**, $\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$ can be presented either symbolically or numerically.
- **P8** might be earned in one or more steps.
- A response of $1.031\cdot 15$ or $[\text{answer from part A}]\cdot 15$ earns both **P8** and **P9**.
- **P9** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

	Model Solution	Scoring	
(a)	Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.		
	$r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (-4.4)}{10 - 7}$	$r''(8.5)$ with supporting work	**1 point**
	$= \frac{0.6}{3} = 0.2$ centimeter per day per day	Units	**1 point**

Scoring notes:

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as cm/day^2).

Total for part (a) 2 points

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

$r(t)$ is twice-differentiable. $\Rightarrow r'(t)$ is differentiable. $\Rightarrow r'(t)$ is continuous. $r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$.	$r'(0) < -6 < r'(3)$ Conclusion using Intermediate Value Theorem	**1 point** **1 point**

Scoring notes:

- To earn the first point, the response must establish that -6 is between $r'(0)$ and $r'(3)$ (or -6.1 and -5). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not state that -6 is between -6.1 and -5. Thus this response does not earn the first point.
- To earn the second point:
  - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
  - The response must have earned the first point.
    - Exception: A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
  - The response must conclude that there is a time t such that $r'(t) = -6$. (A statement of "yes" would be sufficient.)
- To earn the second point, the response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

Total for part (b) 2 points

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

$\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$	Form of right Riemann sum	**1 point**
$= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$	Answer	**1 point**

Scoring notes:

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15, -17.6, -11.4, -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer (-51) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t)\,dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$) earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t)\,dt$ earns no points.
- Units are not required or read in this part.

Total for part (c) 2 points

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$	Product rule	**1 point**
	Chain rule	**1 point**
$\left.\frac{dV}{dt}\right\vert_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70{,}000\pi}{3}$ The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.	Answer	**1 point**

Scoring notes:

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats r or h (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}$ is correct if h is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}\frac{dh}{dt}$ earns any points.
- A response that assumes a functional relationship between r and h (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \rightarrow V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \rightarrow \frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

Total for part (d) 3 points

Total for question 4 9 points

2019

AP® CollegeBoard

AP® Calculus BC

Scoring Guidelines

2019 SCORING GUIDELINES

Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \le 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$3 : \begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

$2 : \begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

2019 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2\,d\theta = 3.534292$

The area of S is 3.534.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} r(\theta)\,d\theta = 1.579933$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \le \theta \le \sqrt{\pi}$ is 1.580 (or 1.579).

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\tan\theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$

$\frac{1}{2}\int_0^{\tan^{-1} m} (r(\theta))^2\,d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\sqrt{\pi}} (r(\theta))^2\,d\theta\right)$

$3 : \begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function applied to } m \text{ as limit of integration} \\ 1 : \text{equation} \end{cases}$

(d) As $k \to \infty$, the circle $r = k\cos\theta$ grows to enclose all points to the right of the y-axis.

$\lim_{k\to\infty} A(k) = \frac{1}{2}\int_0^{\pi/2} (r(\theta))^2\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left(3\sqrt{\theta}\sin\left(\theta^2\right)\right)^2\,d\theta = 3.324$

$2 : \begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$

2019 SCORING GUIDELINES

Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$$3 : \begin{cases} 1 : \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1 : \int_{-2}^{5} f(x)\,dx \\ 1 : \text{answer} \end{cases}$$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$= 2(f(5) - f(3)) + 4(5 - 3)$

$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$

$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$

$= (2f(5) + 20) - (2f(3) + 12)$

$= (2\cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$

$= 2 + 2\sqrt{5}$

$$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$$

(d) $\lim\limits_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$= \frac{10 - 3\cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

2019 SCORING GUIDELINES

Question 4

(a) $V = \pi r^2 h = \pi(1)^2 h = \pi h$

$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second

$$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi\frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

(b) $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}}\cdot\frac{dh}{dt} = -\frac{1}{20\sqrt{h}}\cdot\left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$

Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$$3 : \begin{cases} 1 : \frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}}\cdot\frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$

$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\frac{1}{10}t + C$

$2\sqrt{5} = -\frac{1}{10}\cdot 0 + C \Rightarrow C = 2\sqrt{5}$

$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$

$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2-2x+k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2-2x-8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$
$$= \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| \right]_{x=0}^{x=1}$$
$$= \left(\frac{1}{6}\ln 3 - \frac{1}{6}\ln 3 \right) - \left(\frac{1}{6}\ln 4 - \frac{1}{6}\ln 2 \right) = -\frac{1}{6}\ln 2$$

$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

(c) $$\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-} \left(-\frac{1}{x-1} \Bigg|_{x=0}^{x=b} \right) + \lim_{b\to 1^+} \left(-\frac{1}{x-1} \Bigg|_{x=b}^{x=2} \right)$$
$$= \lim_{b\to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b\to 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\lim\limits_{b\to 1^-} \left(-\dfrac{1}{b-1} \right)$ does not exist, the integral diverges.

$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \dfrac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1+x) + \frac{3}{2}x^2(1)$$
$$= 3 + (3-2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2$$
$$= 3 + x + x^2.$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

(c) $$h(1) = \int_0^1 f(t)\,dt$$
$$\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3 \right) dt$$
$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4 \right]_{t=0}^{t=1}$$
$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4 \right) dt = \left[\frac{9}{20}t^5 \right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left| \frac{9}{20} \right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$

2018

AP® CollegeBoard

AP Calculus BC
Scoring Guidelines

2018 SCORING GUIDELINES

Question 1

(a) $\int_0^{300} r(t)\,dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval $0 \le t \le 300$.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $20 + \int_0^{300} (r(t) - 0.7)\,dt = 20 + \int_0^{300} r(t)\,dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time $t = 300$.

$2 : \begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

(c) Based on part (b), the number of people in line at time $t = 300$ is 80.

The first time t that there are no people in line is

$300 + \frac{80}{0.7} = 414.286$ (or 414.285) seconds.

1 : answer

(d) The total number of people in line at time t, $0 \le t \le 300$, is modeled by $20 + \int_0^t r(x)\,dx - 0.7t$.

$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298,\ t_2 = 166.574719$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

$4 : \begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

2018 SCORING GUIDELINES

Question 2

(a) $p'(25) = -1.179$

At a depth of 25 meters, the density of plankton cells is changing at a rate of -1.179 million cells per cubic meter per meter.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{meaning with units} \end{cases}$

(b) $\int_0^{30} 3p(h)\,dh = 1675.414936$

There are 1675 million plankton cells in the column of water between $h = 0$ and $h = 30$ meters.

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\int_{30}^{K} 3f(h)\,dh$ represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of K meters.

The number of plankton cells, in millions, in the entire column of water is given by $\int_0^{30} 3p(h)\,dh + \int_{30}^{K} 3f(h)\,dh.$

Because $0 \le f(h) \le u(h)$ for all $h \ge 30$,

$3\int_{30}^{K} f(h)\,dh \le 3\int_{30}^{K} u(h)\,dh \le 3\int_{30}^{\infty} u(h)\,dh = 3 \cdot 105 = 315.$

The total number of plankton cells in the column of water is bounded by $1675.415 + 315 = 1990.415 \le 2000$ million.

$3 : \begin{cases} 1 : \text{integral expression} \\ 1 : \text{compares improper integral} \\ 1 : \text{explanation} \end{cases}$

(d) $\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = 757.455862$

The total distance traveled by the boat over the time interval $0 \le t \le 1$ is 757.456 (or 757.455) meters.

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{total distance} \end{cases}$

2018 SCORING GUIDELINES

Question 3

(a) $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx$

$= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx$

$= \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx$

$= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$

$3 : \begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

AP® CALCULUS AB/CALCULUS BC
2018 SCORING GUIDELINES

Question 4

(a) $H'(6) \approx \dfrac{H(7)-H(5)}{7-5} = \dfrac{11-6}{2} = \dfrac{5}{2}$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\dfrac{H(5)-H(3)}{5-3} = \dfrac{6-2}{2} = 2$

Because H is differentiable on $3 \le t \le 5$, H is continuous on $3 \le t \le 5$.

By the Mean Value Theorem, there exists a value c, $3 < c < 5$, such that $H'(c) = 2$.

$2 : \begin{cases} 1 : \dfrac{H(5)-H(3)}{5-3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$

(c) The average height of the tree over the time interval $2 \le t \le 10$ is given by $\dfrac{1}{10-2}\displaystyle\int_2^{10} H(t)\,dt$.

$$\frac{1}{8}\int_2^{10} H(t)\,dt \approx \frac{1}{8}\left(\frac{1.5+2}{2}\cdot 1 + \frac{2+6}{2}\cdot 2 + \frac{6+11}{2}\cdot 2 + \frac{11+15}{2}\cdot 3\right)$$
$$= \frac{1}{8}(65.75) = \frac{263}{32}$$

The average height of the tree over the time interval $2 \le t \le 10$ is $\dfrac{263}{32}$ meters.

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$

(d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x))\cdot\frac{dx}{dt} = \frac{(1+x)100 - 100x\cdot 1}{(1+x)^2}\cdot\frac{dx}{dt} = \frac{100}{(1+x)^2}\cdot\frac{dx}{dt}$$

$$\left.\frac{d}{dt}(G(x))\right|_{x=1} = \frac{100}{(1+1)^2}\cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\dfrac{3}{4}$ meter per year.

$3 : \begin{cases} 2 : \dfrac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0] if no chain rule

AP® CALCULUS BC
2018 SCORING GUIDELINES

Question 5

(a) Area $= \dfrac{1}{2}\displaystyle\int_{\pi/3}^{5\pi/3}\left(4^2 - (3+2\cos\theta)^2\right)d\theta$

$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$

(b) $\dfrac{dr}{d\theta} = -2\sin\theta \Rightarrow \left.\dfrac{dr}{d\theta}\right|_{\theta=\pi/2} = -2$

$r\left(\dfrac{\pi}{2}\right) = 3 + 2\cos\left(\dfrac{\pi}{2}\right) = 3$

$y = r\sin\theta \Rightarrow \dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta$

$x = r\cos\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \dfrac{\pi}{2}$ is $\dfrac{2}{3}$.

— OR —

$y = r\sin\theta = (3+2\cos\theta)\sin\theta \Rightarrow \dfrac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$

$x = r\cos\theta = (3+2\cos\theta)\cos\theta \Rightarrow \dfrac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \dfrac{\pi}{2}$ is $\dfrac{2}{3}$.

$3 : \begin{cases} 1 : \dfrac{dy}{d\theta} = \dfrac{dr}{d\theta}\sin\theta + r\cos\theta \\ \quad\text{or } \dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta - r\sin\theta \\ 1 : \dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ 1 : \text{answer} \end{cases}$

(c) $\dfrac{dr}{dt} = \dfrac{dr}{d\theta}\cdot\dfrac{d\theta}{dt} = -2\sin\theta\cdot\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = \dfrac{dr}{dt}\cdot\dfrac{1}{-2\sin\theta}$

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = 3\cdot\frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$3 : \begin{cases} 1 : \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\cdot\dfrac{d\theta}{dt} \\ 1 : \dfrac{d\theta}{dt} = \dfrac{dr}{dt}\cdot\dfrac{1}{-2\sin\theta} \\ 1 : \text{answer with units} \end{cases}$

AP® CALCULUS BC

2018 SCORING GUIDELINES

Question 6

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2\cdot 3^2} + \frac{x^4}{3\cdot 3^3} - \frac{x^5}{4\cdot 3^4}$.

The general term is $(-1)^{n+1}\frac{x^{n+1}}{n\cdot 3^n}$.

$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$

(b) $\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{(-1)^{n+2}x^{n+2}}{(n+1)\left(3^{n+1}\right)}}{\dfrac{(-1)^{n+1}x^{n+1}}{n\cdot 3^n}}\right| = \lim\limits_{n\to\infty}\left|\frac{-x}{3}\cdot\frac{n}{(n+1)}\right| = \left|\frac{x}{3}\right|$

$\left|\frac{x}{3}\right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x\ln\left(1+\frac{x}{3}\right)$ converges absolutely for $\left|\frac{x}{3}\right| < 1$.

$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

When $x = -3$, the series is $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{(-3)^{n+1}}{n\cdot 3^n} = \sum\limits_{n=1}^{\infty}\frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n\cdot 3^n} = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \le 3$.

$5: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{radius of convergence} \\ 1: \text{considers both endpoints} \\ 1: \text{analysis and interval of convergence} \end{cases}$

— OR —

$5: \begin{cases} 1: \text{radius for } \ln(1+x) \text{ series} \\ 1: \text{substitutes } \frac{x}{3} \\ 1: \text{radius of convergence} \\ 1: \text{considers both endpoints} \\ 1: \text{analysis and interval of convergence} \end{cases}$

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$|P_4(2) - f(2)| < \left|-\frac{2^5}{4\cdot 3^4}\right| = \frac{8}{81}$

$2: \begin{cases} 1: \text{uses fifth-degree term as error bound} \\ 1: \text{answer} \end{cases}$

2017

AP® CollegeBoard

AP Calculus BC

Scoring Guidelines

2017 SCORING GUIDELINES

Question 1

	1 : units in parts (a), (c), and (d)
(a) Volume $= \int_0^{10} A(h)\,dh$ $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ $= 176.3$ cubic feet	$2 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h)\,dh = 101.325338$ The volume is 101.325 cubic feet.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x)\,dx$. $\left.\frac{dV}{dt}\right\rvert_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ $= f(5)\cdot 0.26 = 1.694419$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

2017 SCORING GUIDELINES

Question 2

(a) $\frac{1}{2}\int_0^{\pi/2} (f(\theta))^2\,d\theta = 0.648414$ The area of R is 0.648.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$ — OR — $\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \int_k^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$	$2 : \begin{cases} 1 : \text{integral expression for one region} \\ 1 : \text{equation} \end{cases}$
(c) $w(\theta) = g(\theta) - f(\theta)$ $w_A = \dfrac{\int_0^{\pi/2} w(\theta)\,d\theta}{\frac{\pi}{2} - 0} = 0.485446$ The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.	$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$
(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$ $w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517). $w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.	$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

2017 SCORING GUIDELINES

Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$3 f'(x) = 0 \Rightarrow x = -2, x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because

$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}$.

$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad \text{with explanation} \end{cases}$

2017 SCORING GUIDELINES

Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\dfrac{d^2H}{dt^2} = -\dfrac{1}{4}\dfrac{dH}{dt} = \left(-\dfrac{1}{4}\right)\left(-\dfrac{1}{4}\right)(H - 27) = \dfrac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \dfrac{d^2H}{dt^2} = \dfrac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\dfrac{dG}{(G - 27)^{2/3}} = -dt$

$\displaystyle\int \dfrac{dG}{(G - 27)^{2/3}} = \int (-1)\,dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\dfrac{12 - t}{3}\right)^3$ for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\dfrac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

2017 SCORING GUIDELINES

Question 5

(a) $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2}$ $f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$	$2 : f'(3)$
(b) $f'(x) = \dfrac{-3(4x-7)}{(2x^2-7x+5)^2} = 0 \Rightarrow x = \dfrac{7}{4}$ The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\frac{7}{4}$. f' changes sign from positive to negative at $x = \frac{7}{4}$. Therefore, f has a relative maximum at $x = \frac{7}{4}$.	$2 : \begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \quad \text{with justification} \end{cases}$
(c) $\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \dfrac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\dfrac{2}{2x-5} - \dfrac{1}{x-1}\right)dx$ $= \lim_{b\to\infty}\left[\ln(2x-5) - \ln(x-1)\right]_5^b = \lim_{b\to\infty}\left[\ln\left(\dfrac{2x-5}{x-1}\right)\right]_5^b$ $= \lim_{b\to\infty}\left[\ln\left(\dfrac{2b-5}{b-1}\right) - \ln\left(\dfrac{5}{4}\right)\right] = \ln 2 - \ln\left(\dfrac{5}{4}\right) = \ln\left(\dfrac{8}{5}\right)$	$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$
(d) f is continuous, positive, and decreasing on $[5, \infty)$. The series converges by the integral test since $\int_5^\infty \dfrac{3}{2x^2-7x+5}\,dx$ converges. — OR — $\dfrac{3}{2n^2-7n+5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \ge 5$. Since $\lim_{n\to\infty} \dfrac{\frac{3}{2n^2-7n+5}}{\frac{1}{n^2}} = \dfrac{3}{2}$ and the series $\sum_{n=5}^\infty \dfrac{1}{n^2}$ converges, the series $\sum_{n=5}^\infty \dfrac{3}{2n^2-7n+5}$ converges by the limit comparison test.	2 : answer with conditions

2017 SCORING GUIDELINES

Question 6

(a) $f(0) = 0$ $f'(0) = 1$ $f''(0) = -1(1) = -1$ $f'''(0) = -2(-1) = 2$ $f^{(4)}(0) = -3(2) = -6$ The first four nonzero terms are $0 + 1x + \dfrac{-1}{2!}x^2 + \dfrac{2}{3!}x^3 + \dfrac{-6}{4!}x^4 = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}.$ The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.	$3 : \begin{cases} 1 : f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$
(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^\infty \dfrac{(-1)^{n+1}}{n}$. The series does not converge absolutely because the harmonic series diverges. The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.	2 : converges conditionally with reason
(c) $\int_0^x f(t)\,dt = \int_0^x \left(t - \dfrac{t^2}{2} + \dfrac{t^3}{3} - \dfrac{t^4}{4} + \cdots + \dfrac{(-1)^{n+1}t^n}{n} + \cdots\right)dt$ $= \left[\dfrac{t^2}{2} - \dfrac{t^3}{3\cdot 2} + \dfrac{t^4}{4\cdot 3} - \dfrac{t^5}{5\cdot 4} + \cdots + \dfrac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots\right]_{t=0}^{t=x}$ $= \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{12} - \dfrac{x^5}{20} + \cdots + \dfrac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$	$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$
(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left\|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\|$ is bounded by the magnitude of the first unused term, $\left\|-\dfrac{(1/2)^5}{20}\right\|$. Thus, $\left\|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\| \le \left\|-\dfrac{(1/2)^5}{20}\right\| = \dfrac{1}{32\cdot 20} < \dfrac{1}{500}$.	1 : error bound

AP® Calculus BC
2016 Scoring Guidelines

2016 SCORING GUIDELINES

Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a) $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr^2

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\begin{aligned} \int_0^8 R(t)\,dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $$\begin{aligned} \text{Total} &\approx 50000 + \int_0^8 W(t)\,dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters} \end{aligned}$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

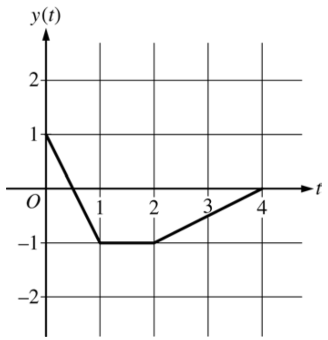
Therefore, the Intermediate Value Theorem guarantees at least one time t, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

2016 SCORING GUIDELINES

Question 2



At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $x(3) = x(0) + \int_0^3 x'(t)\,dt = 5 + 9.377035 = 14.377$

$y(3) = -\frac{1}{2}$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

(b) Slope $= \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$

1 : slope

(c) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969$ (or 9.968)

$2 : \begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

(d) Distance $= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2}\,dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2}\,dt$

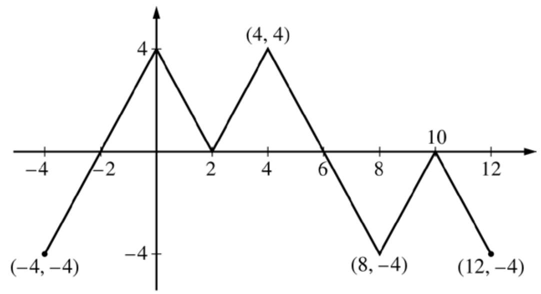
$= 2.237871 + 2.112003 = 4.350$ (or 4.349)

$3 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

2016 SCORING GUIDELINES

Question 3

The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

---

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

$4 : \begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \text{ as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals

2016 SCORING GUIDELINES

Question 4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y.$

(a) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\displaystyle\lim_{x\to-1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$$

$2 : \dfrac{d^2y}{dx^2}$ in terms of x and y

(b) $$\left.\frac{dy}{dx}\right|_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$$

Thus, the graph of f has a relative maximum at the point $(-2, 8)$.

2 : conclusion with justification

(c) $\displaystyle\lim_{x\to-1}(g(x)-2) = 0$ and $\displaystyle\lim_{x\to-1}3(x+1)^2 = 0$

Using L'Hospital's Rule,

$$\lim_{x\to-1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \lim_{x\to-1}\left(\frac{g'(x)}{6(x+1)}\right)$$

$\displaystyle\lim_{x\to-1}g'(x) = 0$ and $\displaystyle\lim_{x\to-1}6(x+1) = 0$

Using L'Hospital's Rule,

$$\lim_{x\to-1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x\to-1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$$

$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

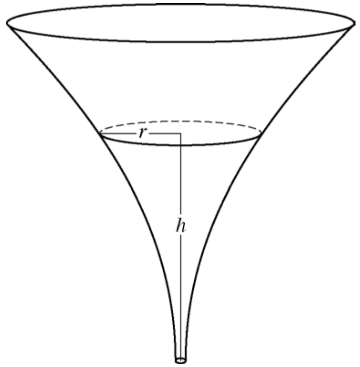
(d) $$h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$$

$$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

2016 SCORING GUIDELINES

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(a) Average radius $$= \frac{1}{10}\int_0^{10}\frac{1}{20}\left(3+h^2\right)dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$$

$$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60} \text{ in}$$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $$= \pi\int_0^{10}\left(\left(\frac{1}{20}\right)\left(3+h^2\right)\right)^2 dh = \frac{\pi}{400}\int_0^{10}\left(9 + 6h^2 + h^4\right)dh$$

$$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$$

$$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40} \text{ in}^3$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $$\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$$

$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{5}\cdot\frac{10}{3} = -\frac{2}{3} \text{ in/sec}$$

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2016 SCORING GUIDELINES

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \ge 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \le 3$.

$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \le \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \le 0.001$$

$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$