

Part B (BC): Graphing calculator not allowed

Question 5

9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

	Model Solution	Scoring
A	Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.	
	$\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y \frac{dy}{dx}$	Product rule Point 1 (P1)
		Chain rule Point 2 (P2)
	$f'(1) = \frac{dy}{dx} \Big _{(x,y)=(1,-1)} = (3 - 1)(-1)^2 = 2$	$f''(1)$ Point 3 (P3)
	$f''(1) = \frac{d^2y}{dx^2} \Big _{(x,y)=(1,-1)} = -(-1)^2 + (3 - 1)(2)(-1)(2) = -9$	

Scoring Notes for Part A

- The expression $\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y$ or $\frac{d^2y}{dx^2} = 6y - y^2 - 2xy$ earns **P1** but not **P2**. Such a response is not eligible for **P3**.
- The expression $\frac{d^2y}{dx^2} = -2y \frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = 6y \left(\frac{dy}{dx} \right) - 2y \left(\frac{dy}{dx} \right)$ earns **P2** but not **P1**. Such a response is eligible for **P3** for a consistent answer of $f''(1) = 4$ or $f''(1) = -8$, respectively, which is found by correctly substituting correct values for x , y , and $\frac{dy}{dx}$.

- A response of $-(-1)^2 + (3 - 1)(2)(-1)(2)$ earns **P1**, **P2**, and **P3** regardless of any subsequent errors in simplification.

- Alternate approach (using separation of variables):

The particular solution for the differential equation that passes through the point $(1, -1)$ is

$$y = \frac{2}{x^2 - 6x + 3}. \text{ Therefore, } \frac{dy}{dx} = \frac{-2(2x - 6)}{(x^2 - 6x + 3)^2}.$$

This response has not yet earned **P1**, **P2**, or **P3**.

- A response that correctly applies the quotient rule (or product rule) and the chain rule to find that $\frac{d^2y}{dx^2} = \frac{-4(x^2 - 6x + 3)^2 + 4(2x - 6)(x^2 - 6x + 3)(2x - 6)}{(x^2 - 6x + 3)^4}$ earns **P1** and **P2** and is eligible to earn **P3** for the correct answer of $f''(1) = -9$.
- A response that correctly applies the quotient rule (or product rule) but does not correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4(x^2 - 6x + 3)^2 + 4(2x - 6)(x^2 - 6x + 3)}{(x^2 - 6x + 3)^4}$) earns **P1**, does not earn **P2**, and is not eligible to earn **P3**.
- A response that does not correctly apply the quotient rule (or product rule) but does correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4}{2(x^2 - 6x + 3)(2x - 6)}$) does not earn **P1**, earns **P2**, and is eligible to earn **P3** for a consistent answer.

B Write the second-degree Taylor polynomial for f about $x = 1$.

$$f'(1) = 2 \text{ and } f''(1) = -9$$

Two terms

Point 4 (P4)

$$P_2(x) = -1 + 2(x - 1) - \frac{9}{2}(x - 1)^2$$

Remaining term

Point 5 (P5)

Scoring Notes for Part B

- **P4** and **P5** can be earned with an answer consistent with incorrect values of $f'(1)$ and $f''(1)$ imported from part A.
- Any terms of degree greater than two or “+...” does not earn **P5**.
- A response of $-1 + 2(x - 1) - \frac{9}{2}(x - 1)^2$ earns **P4** and **P5**, regardless of any subsequent algebraic simplification.
- A response that does not present the polynomial as powers of $(x - 1)$ but instead presents a correct expanded/simplified form of the polynomial (e.g., $-\frac{9}{2}x^2 + 11x - \frac{15}{2}$) earns **P4** but not **P5**.

C The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

$$|f(1.1) - P_2(1.1)| \leq \frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \frac{60}{6} (0.1)^3 = 0.01$$

Form of error bound

Point 6 (P6)

Analysis

Point 7 (P7)

Scoring Notes for Part C

- **P6** is earned for presenting either $\frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3$ or $\frac{60}{6} (0.1)^3$. Subsequent errors in simplification will not earn **P7**.
- To earn **P7**, a response must have earned **P6** and must explicitly connect the error bound with 0.01; for example by communicating Error ≤ 0.01 , Error Bound = 0.01, or equivalent.
- A response that declares the error is equal to 0.01 (or any equivalent form of this value) does not earn **P7**.

D Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

$$f(1.2) \approx f(1) + (1.2 - 1) \cdot \frac{dy}{dx} \Big|_{(1, -1)}$$

$$= -1 + 0.2(2) = -0.6$$

First step of Euler's method

Point 8 (P8)

$$f(1.4) \approx f(1.2) + (1.4 - 1.2) \cdot \frac{dy}{dx} \Big|_{(1.2, -0.6)}$$

$$\approx -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.4704$$

Answer with supporting work

Point 9 (P9)

An approximation of $f(1.4)$ is -0.47 .

Scoring Notes for Part D

- To earn **P8**, a response must demonstrate the first step of Euler's method, with the correct initial condition, correct step size, and correct (or imported) expression for the derivative.
Note: Any subsequent error in simplification or rounding will not affect the scoring for **P8**.
- The two steps of Euler's method may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx} \cdot \Delta x$ (or $\frac{dy}{dx} \cdot 0.2$)
1	-1	0.4
1.2	-0.6	0.1296
1.4	-0.4704	

Note: In the presence of a correct answer, a table does not need to be labeled to earn both **P8** and **P9**. In the presence of no answer or an incorrect answer, such a table must be correctly labeled to earn **P8**.

- A response of $-0.6 + 0.2(3 - 1.2)(-0.6)^2$ earns **P9**, regardless of any subsequent errors in simplification or rounding.
- A response that imports an incorrect value for $f'(1)$ from part A or part B is eligible to earn **P9** with a consistent answer.
- A response may report the final answer as -0.47 , $(1.4, -0.47)$, $-\frac{294}{625}$, or equivalent.