Part B (BC): Graphing calculator not allowed

Question 6 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

	Model Solution	Scoring	
A	Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.		
	$\left\lvert \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}} \right\rvert = \left\lvert \frac{(x-4)(n+1)}{3(n+2)} \right\rvert$	Sets up ratio	**Point 1 (P1)**
	$\lim_{n\to\infty} \left\lvert \frac{(x-4)(n+1)}{3(n+2)} \right\rvert = \left\lvert \frac{x-4}{3} \right\rvert \lim_{n\to\infty} \frac{n+1}{n+2} = \left\lvert \frac{x-4}{3} \right\rvert$	Limit of ratio	**Point 2 (P2)**
	$\left\lvert \frac{x-4}{3} \right\rvert < 1$ when $-3 < x - 4 < 3$. Thus, the series converges when $1 < x < 7$.	Interior of interval of convergence	**Point 3 (P3)**
	When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3}{n+1}$, which converges by the alternating series test.	Considers both endpoints	**Point 4 (P4)**
	When $x = 7$, the series is $\sum_{n=1}^{\infty} \frac{3}{n+1}$, which diverges by limit comparison to the harmonic series. Therefore, the series converges for $1 \le x < 7$.	Analysis and interval of convergence	**Point 5 (P5)**

Scoring Notes for Part A

- **P1** is earned by presenting a correct ratio with or without absolute values. Once earned, **P1** is banked (i.e., subsequent errors in simplification or evaluation do not impact scoring for **P1**). **P2** is not earned if there are any errors in simplification or evaluation of the limit.
- **P1** is earned for ratios mathematically equivalent to any of the following:

$$\frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}, \quad \frac{\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}}{\frac{(x-4)^{n+1}}{(n+1)3^n}}, \quad \frac{(x-4)^{n+1}}{(n+1)3^n} \cdot \frac{n3^{n-1}}{(x-4)^n}, \text{ or } \frac{\frac{(x-4)^{n+1}}{(n+1)3^n}}{\frac{(x-4)^n}{n3^{n-1}}}.$$

- **P1** is also earned for ratios mathematically equivalent to the following reciprocal ratios:

$$\frac{(x-4)^{n+1}}{(n+1)3^n} \cdot \frac{(n+2)3^{n+1}}{(x-4)^{n+2}}, \quad \frac{\frac{(x-4)^{n+1}}{(n+1)3^n}}{\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}}, \quad \frac{(x-4)^n}{n3^{n-1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}, \text{ or } \frac{\frac{(x-4)^n}{n3^{n-1}}}{\frac{(x-4)^{n+1}}{(n+1)3^n}}.$$

  A response that presents any of these reciprocal ratios earns **P1** and is eligible for **P2**, **P3**, and **P4**, but not **P5**.
- A response that does not present a ratio is not eligible for **P2**.
- A response that does not use the absolute value of a ratio can earn both **P1** and **P2**.
- A response that does not use absolute value in computing the limit is eligible for **P3** if the expression $-1 < \frac{x-4}{3} < 1$ or an equivalent inequality is presented.
- A response that presents an incorrect limit of form $\frac{|x-4|}{b}$, where $b > 0$, is eligible for **P3**.

  P3 is then earned for correctly finding an interval with interior $(1, 7)$ or $(4 - b, 4 + b)$.

  Note: $|x - 4| < b$ is not sufficient to earn **P3**. $x - 4 < b$ can earn **P3** if this is resolved to $-b + 4 < x < b + 4$.
- **P4** is earned for considering both endpoints of the correct interval or both endpoints of an incorrect interval that has earned **P3**.
- To earn **P5**, a response must correctly analyze the series at $x = 1$ and $x = 7$, and present the correct interval of convergence. Naming of an appropriate test is sufficient for the analysis at each endpoint. In addition to the tests listed in the model solution, the direct comparison test to an appropriate series or the integral test may also be used.

B	Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.		
	$f'(x) = \frac{(x-4)}{3} + \frac{(x-4)^2}{9} + \frac{(x-4)^3}{27} + \cdots + \frac{(x-4)^n}{3^n} + \cdots$	First three terms	**Point 6 (P6)**
		General term	**Point 7 (P7)**

Scoring Notes for Part B

- **P6** is earned by presenting the first three nonzero terms in a list or as part of a polynomial or series.
- **P7** is earned by identifying the correct general term (either individually or as part of a polynomial or series).

C	The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.		
	The Taylor series for f' is a geometric series with first term $\frac{x-4}{3}$ and common ratio $\frac{x-4}{3}$. $f'(x) = \frac{\frac{x-4}{3}}{1-\frac{x-4}{3}} = \frac{x-4}{3-(x-4)} = \frac{x-4}{7-x}$	Verification	**Point 8 (P8)**

Scoring Notes for Part C

- A response of $f'(x) = \frac{\frac{x-4}{3}}{1-\frac{x-4}{3}}$ is sufficient to earn **P8**.

D	It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.		
	It follows from the work in part A that the interior of the interval of convergence of the Taylor series for f' is $1 < x < 7$. Therefore, $x = 8$ would be outside the interval of convergence of f', and the Taylor series for f' would not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.	Answer with reason	**Point 9 (P9)**

Scoring Notes for Part D

- A response of "no, $x = 8$ is outside the interval of convergence" is sufficient to earn **P9**.
- **P9** can be earned with a response consistent with an incorrect interval of convergence imported from part A.
- Alternate solutions:
  - Because the series for $f'(x)$ is geometric, this converges to $f(x)$ for all values of x such that the common ratio $\frac{x-4}{3}$ is between -1 and 1.

    $-1 < \frac{x-4}{3} < 1 \Rightarrow -3 < x-4 < 3 \Rightarrow 1 < x < 7$
  - Because $x = 8$ is outside the interval $1 < x < 7$, the series for f' does not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.
  - For $x = 8$, the series is given by $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$, which is a geometric series with $r = \frac{4}{3} > 1$. Therefore, the series diverges for $x = 8$.

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General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

Model Solution	Scoring	
(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.		
$\lim_{n\to\infty}\left\lvert\frac{\frac{(-1)^{n+1}x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}}\right\rvert = \lim_{n\to\infty}\left\lvert\frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}}\right\rvert = \lim_{n\to\infty}\left\lvert x^2\left(\frac{2n+1}{2n+3}\right)\right\rvert = \lvert x^2\rvert$	Sets up ratio	**1 point**
$\lvert x^2\rvert < 1$ for $\lvert x\rvert < 1$. The series converges when $-1 < x < 1$.	Identifies interior of interval of convergence	**1 point**
When $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots$.	Considers both endpoints	**1 point**
The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. When $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. Therefore, the interval of convergence is $-1 \le x \le 1$.	Analysis and interval of convergence	**1 point**

Scoring notes:

- A response that includes the substitution error of the form $x^{2(n+1)+1} = x^{2n+3}$ appearing as $x^{2n+1+1} = x^{2n+2}$ in setting up a ratio is eligible for the first 3 points but does not earn the fourth point.
- The first point is earned by presenting a correct ratio with or without absolute values.
- To earn the second point a response must:
  - use the absolute value of the ratio, or resolve the lack of absolute values by concluding $x^2 < 1$ (without any errors), and correctly evaluate the limit of the ratio, including correct limit notation, and
  - identify the interior of the interval of convergence. The response can use either interval notation or the compound inequality $-1 < x < 1$ ($\lvert x\rvert < 1$ is insufficient).
- The only incorrect interval eligible for the third point is $0 < x < 1$. In this case, to earn the third point, the response needs to evaluate the general term at $x = 1$.

Total for part (a) — 4 points

(b) Show that $\left\lvert f\left(\frac{1}{2}\right) - \frac{1}{2}\right\rvert < \frac{1}{10}$. Justify your answer.		
The series for $f\left(\frac{1}{2}\right)$ is an alternating series whose terms decrease in absolute value to 0. The first term of the series for $f\left(\frac{1}{2}\right)$ is $\frac{1}{2}$. Using the alternating series error bound, $f\left(\frac{1}{2}\right)$ differs from $\frac{1}{2}$ by at most the absolute value of the second term of the series. $\left\lvert f\left(\frac{1}{2}\right) - \frac{1}{2}\right\rvert < \left\lvert\frac{(-1)^1\left(\frac{1}{2}\right)^3}{3}\right\rvert = \frac{1}{24} < \frac{1}{10}$	Uses second term Justification	**1 point** **1 point**

Scoring notes:

- The first point is earned by correctly using $x = \frac{1}{2}$ in the second term (listing the second term as part of a polynomial is insufficient). Using $x = \frac{1}{2}$ in any term of degree five or higher does not earn this point.
- To earn the second point a response must:
  - have earned the first point,
  - state that the series is alternating and that its terms decrease to zero, and
  - present the inequality Error $< \frac{1}{24} < \frac{1}{10}$ (or the equivalent).
- A response that states Error $= \frac{1}{24}$ does not earn the second point.

Total for part (b) — 2 points

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$	First four terms	**1 point**
	General term	**1 point**

Scoring notes:

- The first point is earned by presenting the first four nonzero terms in a list or as part of a polynomial or series.
- The second point is earned by identifying the general term (either individually or as part of a polynomial or series).
- Read "$=$" as "$\approx$" as necessary.

Total for part (c) 2 points

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

$f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \cdots$ $f'\left(\frac{1}{6}\right)$ is a geometric series with $a = 1$ and $r = -\frac{1}{36}$. $f'\left(\frac{1}{6}\right) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{36}\right)} = \frac{1}{\frac{37}{36}} = \frac{36}{37}$	Answer	**1 point**

Scoring notes:

- The result from part (c) must be geometric in order to be eligible for this point.
- If a response imports an incorrect geometric series from part (c), this point is earned only for a consistent answer.

Total for part (d) 1 point

Total for question 6 9 points