Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3-x)y^2$ with initial condition $f(1) = -1$.

	Model Solution	Scoring	
A	Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.		
	$\frac{d^2y}{dx^2} = -y^2 + (3-x)2y\frac{dy}{dx}$	Product rule	**Point 1 (P1)**
		Chain rule	**Point 2 (P2)**
	$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,y)=(1,-1)} = (3-1)(-1)^2 = 2$ $f''(1) = \left.\frac{d^2y}{dx^2}\right\|_{(x,y)=(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -9$	$f''(1)$	**Point 3 (P3)**

Scoring Notes for Part A

- The expression $\frac{d^2y}{dx^2} = -y^2 + (3-x)2y$ or $\frac{d^2y}{dx^2} = 6y - y^2 - 2xy$ earns **P1** but not **P2**. Such a response is not eligible for **P3**.
- The expression $\frac{d^2y}{dx^2} = -2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = 6y\left(\frac{dy}{dx}\right) - 2y\left(\frac{dy}{dx}\right)$ earns **P2** but not **P1**. Such a response is eligible for **P3** for a consistent answer of $f''(1) = 4$ or $f''(1) = -8$, respectively, which is found by correctly substituting correct values for x, y, and $\frac{dy}{dx}$.
- A response of $-(-1)^2 + (3-1)(2)(-1)(2)$ earns **P1**, **P2**, and **P3** regardless of any subsequent errors in simplification.
- Alternate approach (using separation of variables):

  The particular solution for the differential equation that passes through the point $(1,-1)$ is $y = \frac{2}{x^2-6x+3}$. Therefore, $\frac{dy}{dx} = \frac{-2(2x-6)}{\left(x^2-6x+3\right)^2}$.

  This response has not yet earned **P1**, **P2**, or **P3**.
  - A response that correctly applies the quotient rule (or product rule) and the chain rule to find that $\frac{d^2y}{dx^2} = \frac{-4\left(x^2-6x+3\right)^2 + 4(2x-6)\left(x^2-6x+3\right)(2x-6)}{\left(x^2-6x+3\right)^4}$ earns **P1** and **P2** and is eligible to earn **P3** for the correct answer of $f''(1) = -9$.
  - A response that correctly applies the quotient rule (or product rule) but does not correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4\left(x^2-6x+3\right)^2 + 4(2x-6)\left(x^2-6x+3\right)}{\left(x^2-6x+3\right)^4}$) earns **P1**, does not earn **P2**, and is not eligible to earn **P3**.
  - A response that does not correctly apply the quotient rule (or product rule) but does correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4}{2\left(x^2-6x+3\right)(2x-6)}$) does not earn **P1**, earns **P2**, and is eligible to earn **P3** for a consistent answer.

B	Write the second-degree Taylor polynomial for f about $x = 1$.		
	$f'(1) = 2$ and $f''(1) = -9$ $P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$	Two terms Remaining term	**Point 4 (P4)** **Point 5 (P5)**

Scoring Notes for Part B

- **P4** and **P5** can be earned with an answer consistent with incorrect values of $f'(1)$ and $f''(1)$ imported from part A.
- Any terms of degree greater than two or "$+\cdots$" does not earn **P5**.
- A response of $-1 + 2(x-1) - \frac{9}{2}(x-1)^2$ earns **P4** and **P5**, regardless of any subsequent algebraic simplification.
- A response that does not present the polynomial as powers of $(x-1)$ but instead presents a correct expanded/simplified form of the polynomial (e.g., $-\frac{9}{2}x^2 + 11x - \frac{15}{2}$) earns **P4** but not **P5**.

C	The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $\lvert f'''(x)\rvert \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.		
	$\lvert f(1.1) - P_2(1.1)\rvert \le \frac{\max_{1\le x\le 1.1}\lvert f'''(x)\rvert}{3!}\lvert 1.1-1\rvert^3 \le \frac{60}{6}(0.1)^3 = 0.01$	Form of error bound Analysis	**Point 6 (P6)** **Point 7 (P7)**

Scoring Notes for Part C

- **P6** is earned for presenting either $\frac{\max_{1\le x\le 1.1}\lvert f'''(x)\rvert}{3!}\lvert 1.1-1\rvert^3$ or $\frac{60}{6}(0.1)^3$. Subsequent errors in simplification will not earn **P7**.
- To earn **P7**, a response must have earned **P6** and must explicitly connect the error bound with 0.01; for example by communicating $\text{Error} \le 0.01$, $\text{Error Bound} = 0.01$, or equivalent.
- A response that declares the error is equal to 0.01 (or any equivalent form of this value) does not earn **P7**.

D	Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.		
	$f(1.2) \approx f(1) + (1.2-1)\cdot \left.\frac{dy}{dx}\right\rvert_{(1,\,-1)}$ $= -1 + 0.2(2) = -0.6$ $f(1.4) \approx f(1.2) + (1.4-1.2)\cdot \left.\frac{dy}{dx}\right\rvert_{(1.2,\,-0.6)}$ $\approx -0.6 + 0.2(3-1.2)(-0.6)^2 = -0.4704$ An approximation of $f(1.4)$ is -0.47.	First step of Euler's method Answer with supporting work	**Point 8 (P8)** **Point 9 (P9)**

Scoring Notes for Part D

- To earn **P8**, a response must demonstrate the first step of Euler's method, with the correct initial condition, correct step size, and correct (or imported) expression for the derivative.

  Note: Any subsequent error in simplification or rounding will not affect the scoring for **P8**.
- The two steps of Euler's method may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx}\cdot\Delta x$ (or $\frac{dy}{dx}\cdot 0.2$)
1	-1	0.4
1.2	-0.6	0.1296
1.4	-0.4704	

  Note: In the presence of a correct answer, a table does not need to be labeled to earn both **P8** and **P9**. In the presence of no answer or an incorrect answer, such a table must be correctly labeled to earn **P8**.
- A response of $-0.6 + 0.2(3-1.2)(-0.6)^2$ earns **P9**, regardless of any subsequent errors in simplification or rounding.
- A response that imports an incorrect value for $f'(1)$ from part A or part B is eligible to earn **P9** with a consistent answer.
- A response may report the final answer as -0.47, $(1.4, -0.47)$, $-\frac{294}{625}$, or equivalent.

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f(x^2)$, and $f'''(x) = -2x \cdot f'(x^2)$.

Model Solution	Scoring	

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

$f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) f''(x^2) \cdot 2x$	Form of product rule	**1 point**
	$f^{(4)}(x)$	**1 point**
$f''(0) = -f(0) = -2$ $f'''(0) = -2(0) \cdot f'(0) = 0$ $f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$	Two terms of polynomial	**1 point**
The fourth-degree Taylor polynomial for f about $x = 0$ is $T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4$ $= 2 + 3x - x^2 - \frac{1}{4}x^4$	Remaining terms	**1 point**

Scoring notes:

- The first point is earned for a correct fourth derivative or for $f^{(4)}(x) = -2 \cdot f'(x^2) + (-2x) f''(x^2)$.
- The second point is earned only for a completely correct expression for $f^{(4)}(x)$.
- A response that earns the first point but not the second may evaluate the presented expression for $f^{(4)}(x)$ at $x = 0$ and use the consistent nonzero value in computing the coefficient of x^4 in the fourth-degree Taylor polynomial.
- A polynomial that includes a nonzero third-degree term, any terms of degree greater than four, or $+\ldots$ does not earn the fourth point.

Total for part (a) 4 points

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

By the Lagrange error bound, $\left|T_4(0.1) - f(0.1)\right| \le \frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5$ $\le \frac{15}{120} \cdot \frac{1}{10^5} \le \frac{1}{10^5}$	Form of error bound	**1 point**
	Shows $\left|\text{Error}\right| \le \frac{1}{10^5}$	**1 point**

Scoring notes:

- The first point is earned for $\frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$. Subsequent errors in simplification will not earn the second point.
- To earn the second point a response must communicate the inequality $\text{Error} \le \frac{15}{5!} \cdot (0.1)^5 \le \frac{1}{10^5}$.
- A response that states $\text{Error} = \frac{15}{5!} \cdot (0.1)^5$ or $\text{Error} = \frac{1}{10^5}$ does not earn the second point.

Total for part (b) 2 points

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

$g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$	$g''(x)$	**1 point**
$g'(0) = e^0 \cdot f(0) = 2$ $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0) = 2 + 3 = 5$	First two terms of polynomial	**1 point**
The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2.$	Taylor polynomial	**1 point**

Scoring notes:

- The first point is earned for $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$, $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0)$, or $g''(0) = f(0) + f'(0)$.
- A presented polynomial of the form $4 + 2x + ax^2$ earns the second point with or without any supporting work for the first two terms.
- A response that earned neither the first nor the second point only earns the third point for a polynomial of the form $a + bx + \frac{c}{2}x^2$, where $c \ne 0$ is declared to be $g''(0)$.

- A polynomial that includes any terms of degree greater than two, or $+\ldots,$ does not earn the third point.
- Alternate solution:

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

$$e^x f(x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(2 + 3x - x^2 + \cdots\right) = 2 + 5x + \cdots$$

$$g(x) = \int e^x f(x)\,dx = C + 2x + \frac{5}{2}x^2 + \cdots$$

$$g(0) = 4 \Rightarrow C = 4$$

$$g(x) \approx 4 + 2x + \frac{5}{2}x^2$$

  - A response that is using this alternate solution method earns the first point for $e^x f(x) = 2 + 5x + \cdots,$ the second point for any two correct terms in a second-degree polynomial, and the third point for a completely correct second-degree Taylor polynomial with supporting work.
  - Note: There is not enough information to conclude that $f(x)$ is equal to its Maclaurin series on its interval of convergence. The second and third lines of the alternate solution are being accepted as identifications of the Maclaurin series for $e^x f(x)$ and $g(x),$ respectively.

Total for part (c)	**3 points**
Total for question 6	**9 points**