

Part B (BC): Graphing calculator not allowed**Question 6****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

Model Solution	Scoring
(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.	
At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.	Considers $\frac{(n+1)6^n}{n^2 6^n}$ 1 point
Because $\frac{n+1}{n^2} > \frac{1}{n}$ for all $n \geq 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the comparison test.	Answer with reason 1 point

Scoring notes:

- To earn the first point using either the comparison or limit comparison test, a response must consider the term $\frac{(n+1)6^n}{n^2 6^n}$. This could be shown by considering the term $\frac{n+1}{n^2}$, either individually or as part of a sum.
- To earn the second point using the comparison test a response must demonstrate that the terms $\frac{n+1}{n^2}$ are larger than the terms in a divergent series.
 - “ $\frac{n+1}{n^2} > \frac{1}{n}$, diverges” earns both points.
 - The response does not need to use the term “comparison test,” but the response cannot declare use of an incorrect test.
- Alternate solution (limit comparison test):

At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.

Because $\lim_{n \rightarrow \infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2 + n}{n^2} = 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series

$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the limit comparison test.

- To earn the second point using the limit comparison test, a response must correctly write the limit of the ratio of the terms in the given series to the terms of a divergent series and demonstrate that the limit of this ratio is 1.
- The response does not need to use the term “limit comparison test,” but the response cannot declare use of an incorrect test.

Total for part (a) 2 points

- (b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} \left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $|f(-3) - S_3| < \frac{1}{50}$.

$f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2} \left(-\frac{1}{2}\right)^n$ is an alternating series with terms that decrease in magnitude to 0.

By the alternating series error bound, $\sum_{n=1}^3 \frac{n+1}{n^2} \left(-\frac{1}{2}\right)^n = -\frac{125}{144}$

approximates $f(-3)$ with error of at most

$$\left| \frac{4+1}{4^2} \left(-\frac{1}{2}\right)^4 \right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}.$$

Thus, $|f(-3) - S_3| < \frac{1}{50}$.

Uses fourth term **1 point**

Verification **1 point**

Scoring notes:

- The first point is earned for correctly using $x = -3$ in the fourth term. (Listing the fourth term as part of a polynomial is not sufficient.) Using $x = -3$ in any term of degree five or higher does not earn this point.
- The expression $\frac{4+1}{4^2} \left(-\frac{1}{2}\right)^4$ earns the first point, but just $\frac{5}{256}$ does not earn the first point.
- A response including the expression $\frac{4+1}{4^2} \left(-\frac{1}{2}\right)^4$ that is subsequently simplified incorrectly earns the first point but not the second.
- To earn the second point the response must state that the series for $f(-3)$ is alternating or that the alternating series error bound is being used.
 - A response of just “Error $\leq \frac{4+1}{4^2} \left(-\frac{1}{2}\right)^4 < \frac{1}{50}$ ” (or any equivalent mathematical expression) earns both points, provided it is accompanied by an indication that the series is alternating.
- A response that declares the error is equal to $\frac{5}{256}$ (or any equivalent form of this value) does not earn the second point.

Total for part (b) 2 points

- (c) Find the general term of the Maclaurin series for f' , the derivative of f . Find the radius of convergence of the Maclaurin series for f' .

The general term of the Maclaurin series for f' is $\frac{(n+1)nx^{n-1}}{n^2 6^n} = \frac{(n+1)x^{n-1}}{n \cdot 6^n}.$	General term 1 point
Because the radius of convergence of the Maclaurin series for f is 6, the radius of convergence of the Maclaurin series for f' is also 6.	Radius 1 point

Scoring notes:

- A response of $\frac{(n+1)nx^{n-1}}{n^2 6^n}$ earns the first point. Any expression mathematically equivalent to this also earns the first point.
- The response need not simplify $\frac{(n+1)nx^{n-1}}{n^2 6^n}$, but any presented simplification must be correct in order to earn the first point.
- The second point is earned only for a supported answer of 6. The second point can be earned without the first.
- Alternate solution for second point (ratio test):

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(n+2)x^n}{(n+1)6^{n+1}}}{\frac{(n+1)x^{n-1}}{n \cdot 6^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(n+2)}{(n+1)^2} \cdot \frac{x}{6} \right| = \frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

Therefore, the radius of convergence of the Maclaurin series for f' is 6.

- Alternate solution for second point (root test):

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n+1)|x|^{n-1}}{n \cdot 6^n}} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{1/n} \cdot |x|^{-1/n} \cdot \frac{|x|}{6} = 1 \cdot 1 \cdot \frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

Therefore, the radius of convergence of the Maclaurin series for f' is 6.

Total for part (c) 2 points

- (d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g .

$\left \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}} \right $	Sets up ratio	1 point
$\lim_{n \rightarrow \infty} \left \frac{(n+2)n^2}{(n+1)^3} \cdot \left \frac{x^2}{3} \right \right = \left \frac{x^2}{3} \right $	Limit	1 point
$\left \frac{x^2}{3} \right < 1 \Rightarrow x^2 < 3 \Rightarrow x < \sqrt{3}$ The radius of convergence of g is $\sqrt{3}$.	Radius of convergence	1 point

Scoring notes:

- The first point is earned by presenting a correct ratio with or without absolute values. Once earned, this point cannot be lost. Any errors in simplification or evaluation of the limit will not earn the second point.
- The first point is earned for ratios mathematically equivalent to any of the following:

$$\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}, \quad \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}, \quad \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n-1)^2 3^{n-1}}{nx^{2n-2}}, \quad \text{or} \quad \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n-1)^2 3^{n-1}}{nx^{2n-2}}.$$

- The first point is also earned for ratios mathematically equivalent to the following reciprocal ratios:

$$\frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n+1)^2 3^{n+1}}{(n+2)x^{2n+2}}, \quad \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n+1)^2 3^{n+1}}{(n+2)x^{2n+2}}, \quad \frac{nx^{2n-2}}{(n-1)^2 3^{n-1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}, \quad \text{or} \quad \frac{nx^{2n-2}}{(n-1)^2 3^{n-1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}.$$

Responses including any of these reciprocal ratios can earn the second point for using limit notation to correctly find a limit of the absolute value of their ratio to be $\left| \frac{3}{x^2} \right|$. Such responses earn the third point only for a final answer of $\sqrt{3}$ with a valid explanation for reporting the reciprocal of $\frac{1}{\sqrt{3}}$.

- To earn the second point a response must use the ratio and correctly evaluate the limit of the ratio, using correct limit notation.
- The third point is earned only for an answer of $\sqrt{3}$ with supporting work.

Total for part (d) 3 points

Total for question 6 9 points

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The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

Model Solution	Scoring
(a) Using the ratio test, find the interval of convergence of the power series for f . Justify your answer.	
$\lim_{n \rightarrow \infty} \left \frac{\frac{(-1)^{n+1} x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right = \lim_{n \rightarrow \infty} \left \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{2n+1}{2n+3} \right = \lim_{n \rightarrow \infty} \left x^2 \left(\frac{2n+1}{2n+3} \right) \right = x^2 $	Sets up ratio 1 point
$ x^2 < 1$ for $ x < 1$. The series converges when $-1 < x < 1$.	Identifies interior of interval of convergence 1 point
When $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test.	Considers both endpoints 1 point
When $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test.	Analysis and interval of convergence 1 point
Therefore, the interval of convergence is $-1 \leq x \leq 1$.	

Scoring notes:

- A response that includes the substitution error of the form $x^{2(n+1)+1} = x^{2n+3}$ appearing as $x^{2n+1+1} = x^{2n+2}$ in setting up a ratio is eligible for the first 3 points but does not earn the fourth point.
- The first point is earned by presenting a correct ratio with or without absolute values.
- To earn the second point a response must:
 - use the absolute value of the ratio, or resolve the lack of absolute values by concluding $x^2 < 1$ (without any errors), and correctly evaluate the limit of the ratio, including correct limit notation, and
 - identify the interior of the interval of convergence. The response can use either interval notation or the compound inequality $-1 < x < 1$ ($|x| < 1$ is insufficient).
- The only incorrect interval eligible for the third point is $0 < x < 1$. In this case, to earn the third point, the response needs to evaluate the general term at $x = 1$.

Total for part (a) 4 points

- (b) Show that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. Justify your answer.

The series for $f\left(\frac{1}{2}\right)$ is an alternating series whose terms decrease in absolute value to 0. The first term of the series for $f\left(\frac{1}{2}\right)$ is $\frac{1}{2}$. Using the alternating series error bound, $f\left(\frac{1}{2}\right)$ differs from $\frac{1}{2}$ by at most the absolute value of the second term of the series.

$$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \left|\frac{(-1)^1\left(\frac{1}{2}\right)^3}{3}\right| = \frac{1}{24} < \frac{1}{10}$$

Uses second term **1 point**Justification **1 point****Scoring notes:**

- The first point is earned by correctly using $x = \frac{1}{2}$ in the second term (listing the second term as part of a polynomial is insufficient). Using $x = \frac{1}{2}$ in any term of degree five or higher does not earn this point.
- To earn the second point a response must:
 - have earned the first point,
 - state that the series is alternating and that its terms decrease to zero, and
 - present the inequality $\text{Error} < \frac{1}{24} < \frac{1}{10}$ (or the equivalent).
- A response that states $\text{Error} = \frac{1}{24}$ does not earn the second point.

Total for part (b) 2 points

- (c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$	First four terms	1 point
	General term	1 point

Scoring notes:

- The first point is earned by presenting the first four nonzero terms in a list or as part of a polynomial or series.
- The second point is earned by identifying the general term (either individually or as part of a polynomial or series).
- Read “=” as “ $\approx$ ” as necessary.

Total for part (c) 2 points

- (d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

$f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \cdots$ $f'\left(\frac{1}{6}\right) \text{ is a geometric series with } a = 1 \text{ and } r = -\frac{1}{36}.$ $f'\left(\frac{1}{6}\right) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{36}\right)} = \frac{1}{\frac{37}{36}} = \frac{36}{37}$	Answer	1 point
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Scoring notes:

- The result from part (c) must be geometric in order to be eligible for this point.
- If a response imports an incorrect geometric series from part (c), this point is earned only for a consistent answer.

Total for part (d) 1 point

Total for question 6 9 points

Part B (BC): Graphing calculator not allowed**Question 6****9 points****General Scoring Notes**

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Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3} \text{ on its interval of convergence.}$$

	Model Solution	Scoring
(a)	State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	
	e^{-x} is positive, decreasing, and continuous on the interval $[0, \infty)$.	Conditions 1 point
	To use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, show that $\int_0^{\infty} e^{-x} dx$ is finite (converges).	Improper integral 1 point
	$\int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} \left(-e^{-x} \Big _0^b \right) = \lim_{b \rightarrow \infty} \left(-e^{-b} + e^0 \right) = 1$ Because the integral $\int_0^{\infty} e^{-x} dx$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	Evaluation 1 point

Scoring notes:

- To earn the first point a response must list all three conditions: e^{-x} is positive, decreasing, and continuous.
- The second point is earned for correctly writing the improper integral or for presenting a correct limit equivalent to the improper integral (for example, $\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx$).
- To earn the third point a response must correctly use limit notation to evaluate the improper integral, find an evaluation of e^0 (or 1), and conclude that the integral converges or that the series converges.
- If an incorrect lower limit of 1 is used in the improper integral, then the second point is not earned. In this case, if the correct limit ($1/e$) is presented, then the response is eligible for the third point.
- If the response only relies on using a geometric series approach, then no points are earned [0-0-0].
- A response that presents an evaluation with ∞ , such as $e^{-\infty} = 0$, does not earn the third point.

Total for part (a) 3 points

- (b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

$\lim_{n \rightarrow \infty} \frac{\frac{1}{e^n}}{\left \frac{(-1)^n}{2e^n + 3} \right } = \lim_{n \rightarrow \infty} \frac{2e^n + 3}{e^n} = 2$	Sets up limit comparison 1 point
The limit exists and is positive. Therefore, because the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty} \left \frac{(-1)^n}{2e^n + 3} \right$ converges by the limit comparison test. Thus, the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.	Explanation 1 point

Scoring notes:

- The first point is earned for setting up the limit comparison, with or without absolute values. Limit notation is required to earn this point.
- The reciprocal of the given ratio is an acceptable alternative; the limit in this case is $1/2$.
- The second point cannot be earned without the use of absolute value symbols, which can occur explicitly or implicitly (e.g., a response might set up the limit comparison initially as
$$\lim_{n \rightarrow \infty} \frac{\frac{1}{e^n}}{\frac{1}{2e^n + 3}}).$$
- Earning the second point requires correctly evaluating the limit and noting that the limit is a positive number. For example, $L = 2 > 0$ or $L = 1/2 > 0$. Therefore, comparing the limit L to 1 does not earn the explanation point.
- A response does not have to repeat that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.
- A response that draws a conclusion based only on the sequence (such as $\frac{1}{e^n}$) without referencing a series does not earn the second point.
- If the response does not explicitly use the limit comparison test, then no points are earned in this part.
- A response cannot earn the second point for just concluding that “the series” converges absolutely because there are multiple series in this part of the problem. The response must specify that the series $g(1)$ or $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Total for part (b) 2 points

(c) Determine the radius of convergence of the Maclaurin series for g .

$\left \frac{(-1)^{n+1} x^{n+1}}{2e^{n+1} + 3} \cdot \frac{2e^n + 3}{(-1)^n x^n} \right = \left \frac{(2e^n + 3)x^{n+1}}{(2e^{n+1} + 3)x^n} \right = \frac{2e^n + 3}{2e^{n+1} + 3} x $	Sets up ratio	1 point
$\lim_{n \rightarrow \infty} \frac{2e^n + 3}{2e^{n+1} + 3} x = \frac{1}{e} x $	Computes limit of ratio	1 point
$\frac{1}{e} x < 1 \Rightarrow x < e$ The radius of convergence is $R = e$.	Answer	1 point

Scoring notes:

- The first point is earned for $\frac{(-1)^{n+1} x^{n+1}}{2e^{n+1} + 3} \cdot \frac{2e^n + 3}{(-1)^n x^n}$ or the equivalent. Once earned, this point cannot be lost.
- The second point cannot be earned without the first point.
- To be eligible for the third point the response must have found a limit for a presented ratio such that the limiting value of the coefficient on $|x|$ is finite and not 0. The third point is earned for setting up an inequality such that the limit is less than 1, solving for $|x|$, and interpreting the result to find the radius of convergence.
- The radius of convergence must be explicitly presented, for example, $R = e$. The third point cannot be earned by presenting an interval, for example $-e < x < e$, with no identification of the radius of convergence.

Total for part (c) 3 points

- (d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ decrease in magnitude to 0.

The alternating series error bound for the error of the approximation is the absolute value of the third term of the series.

$$\text{Error} \leq \left| \frac{(-1)^2}{2e^2 + 3} \right| = \frac{1}{2e^2 + 3}$$

Answer

1 point

Scoring notes:

- A response of $\frac{1}{2e^2 + 3}$ earns this point.

Total for part (d)

1 point

Total for question 6

9 points

2019

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Scoring Guidelines

2019 SCORING GUIDELINES

Question 1

(a) $\int_0^5 E(t) dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b) $\frac{1}{5-0} \int_0^5 L(t) dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

- (c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$$E(t) - L(t) = 0 \Rightarrow t = 6.20356$$

$E(t) - L(t) > 0$ for $0 \leq t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \leq 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$$3 : \begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$$A(t) = \int_0^t (E(s) - L(s)) ds$$

$$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

$$2 : \begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$$

2019 SCORING GUIDELINES

Question 2

$$(a) \quad \frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = 3.534292$$

The area of S is 3.534.

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

$$(b) \quad \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.579933$$

The average distance from the origin to a point on the curve $r = r(\theta)$ for $0 \leq \theta \leq \sqrt{\pi}$ is 1.580 (or 1.579).

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

$$(c) \quad \tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$$

$$\frac{1}{2} \int_0^{\tan^{-1} m} (r(\theta))^2 d\theta = \frac{1}{2} \left(\frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta \right)$$

$$3 : \begin{cases} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \text{applied to } m \text{ as limit of} \\ \text{integration} \\ 1 : \text{equation} \end{cases}$$

(d) As $k \rightarrow \infty$, the circle $r = k \cos \theta$ grows to enclose all points to the right of the y -axis.

$$2 : \begin{cases} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{cases}$$

$$\begin{aligned} \lim_{k \rightarrow \infty} A(k) &= \frac{1}{2} \int_0^{\pi/2} (r(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} (3\sqrt{\theta} \sin(\theta^2))^2 d\theta = 3.324 \end{aligned}$$

2019 SCORING GUIDELINES

Question 3

$$\begin{aligned}
 \text{(a)} \quad \int_{-6}^5 f(x) \, dx &= \int_{-6}^{-2} f(x) \, dx + \int_{-2}^5 f(x) \, dx \\
 &\Rightarrow 7 = \int_{-6}^{-2} f(x) \, dx + 2 + \left(9 - \frac{9\pi}{4}\right) \\
 &\Rightarrow \int_{-6}^{-2} f(x) \, dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_3^5 (2f'(x) + 4) \, dx &= 2 \int_3^5 f'(x) \, dx + \int_3^5 4 \, dx \\
 &= 2(f(5) - f(3)) + 4(5 - 3) \\
 &= 2(0 - (3 - \sqrt{5})) + 8 \\
 &= 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5}
 \end{aligned}$$

— OR —

$$\begin{aligned}
 \int_3^5 (2f'(x) + 4) \, dx &= [2f(x) + 4x]_{x=3}^{x=5} \\
 &= (2f(5) + 20) - (2f(3) + 12) \\
 &= (2 \cdot 0 + 20) - (2(3 - \sqrt{5}) + 12) \\
 &= 2 + 2\sqrt{5}
 \end{aligned}$$

$$\text{(c)} \quad g'(x) = f(x) = 0 \Rightarrow x = -1, x = \frac{1}{2}, x = 5$$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$$\begin{aligned}
 \text{(d)} \quad \lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} &= \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \\
 &= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}
 \end{aligned}$$

$$3 : \begin{cases} 1 : \int_{-6}^5 f(x) \, dx = \int_{-6}^{-2} f(x) \, dx + \int_{-2}^5 f(x) \, dx \\ 1 : \int_{-2}^5 f(x) \, dx \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$$

$$1 : \text{answer}$$

2019 SCORING GUIDELINES

Question 4

(a) $V = \pi r^2 h = \pi(1)^2 h = \pi h$
 $\left. \frac{dV}{dt} \right|_{h=4} = \pi \left. \frac{dh}{dt} \right|_{h=4} = \pi \left(-\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5}$ cubic feet per second

$$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

(b) $\frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$
 Because $\frac{d^2 h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$$3 : \begin{cases} 1 : \frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$
 $\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt$
 $2\sqrt{h} = -\frac{1}{10} t + C$
 $2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$
 $2\sqrt{h} = -\frac{1}{10} t + 2\sqrt{5}$
 $h(t) = \left(-\frac{1}{20} t + \sqrt{5} \right)^2$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration

2019 SCORING GUIDELINES

Question 5

$$(a) \quad f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2}$$

$$f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

$$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$$

$$(b) \quad \frac{1}{x^2-2x-8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$\Rightarrow 1 = A(x+2) + B(x-4)$$

$$\Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}$$

$$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$$

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx \\ &= \left[\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1} \\ &= \left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \end{aligned}$$

$$\begin{aligned} (c) \quad \int_0^2 \frac{1}{x^2-2x+1} dx &= \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right) \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right) \end{aligned}$$

Because $\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$$

2019 SCORING GUIDELINES

Question 6

(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-2}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{2}{6}x^3.$$

(b) The first three nonzero terms of the Maclaurin series for e^x are

$$1 + x + \frac{1}{2!}x^2.$$

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} 3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) \\ = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 \\ = 3 + x + x^2. \end{aligned}$$

(c) $h(1) = \int_0^1 f(t) dt$

$$\begin{aligned} &\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt \\ &= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1} \\ &= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48} \end{aligned}$$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \leq \left|\frac{9}{20}\right| = 0.45$$

$$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$$

$$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$$

$$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : \text{uses fourth-degree term} \\ \quad \text{of Maclaurin series for } f \\ 1 : \text{uses first omitted term} \\ \quad \text{of series for } h(1) \\ 1 : \text{error bound} \end{cases}$$

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AP Calculus BC

Scoring Guidelines

2018 SCORING GUIDELINES

Question 1

(a) $\int_0^{300} r(t) dt = 270$

According to the model, 270 people enter the line for the escalator during the time interval $0 \leq t \leq 300$.

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $20 + \int_0^{300} (r(t) - 0.7) dt = 20 + \int_0^{300} r(t) dt - 0.7 \cdot 300 = 80$

According to the model, 80 people are in line at time $t = 300$.

2 : $\begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$

(c) Based on part (b), the number of people in line at time $t = 300$ is 80.

The first time t that there are no people in line is

$$300 + \frac{80}{0.7} = 414.286 \text{ (or 414.285) seconds.}$$

1 : answer

(d) The total number of people in line at time t , $0 \leq t \leq 300$, is modeled by

$$20 + \int_0^t r(x) dx - 0.7t.$$

$$r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$$

4 : $\begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

2018 SCORING GUIDELINES

Question 2

(a) $p'(25) = -1.179$

At a depth of 25 meters, the density of plankton cells is changing at a rate of -1.179 million cells per cubic meter per meter.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{meaning with units} \end{cases}$$

(b) $\int_0^{30} 3p(h) \, dh = 1675.414936$

There are 1675 million plankton cells in the column of water between $h = 0$ and $h = 30$ meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

(c) $\int_{30}^K 3f(h) \, dh$ represents the number of plankton cells, in millions, in the column of water from a depth of 30 meters to a depth of K meters.

The number of plankton cells, in millions, in the entire column of water is given by $\int_0^{30} 3p(h) \, dh + \int_{30}^K 3f(h) \, dh$.

Because $0 \leq f(h) \leq u(h)$ for all $h \geq 30$,

$$3 \int_{30}^K f(h) \, dh \leq 3 \int_{30}^K u(h) \, dh \leq 3 \int_{30}^{\infty} u(h) \, dh = 3 \cdot 105 = 315.$$

The total number of plankton cells in the column of water is bounded by $1675.415 + 315 = 1990.415 \leq 2000$ million.

$$3 : \begin{cases} 1 : \text{integral expression} \\ 1 : \text{compares improper integral} \\ 1 : \text{explanation} \end{cases}$$

(d) $\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = 757.455862$

The total distance traveled by the boat over the time interval $0 \leq t \leq 1$ is 757.456 (or 757.455) meters.

$$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{total distance} \end{cases}$$

2018 SCORING GUIDELINES

Question 3

$$\begin{aligned} \text{(a)} \quad f(-5) &= f(1) + \int_1^{-5} g(x) \, dx = f(1) - \int_{-5}^1 g(x) \, dx \\ &= 3 - \left(-9 - \frac{3}{2} + 1 \right) = 3 - \left(-\frac{19}{2} \right) = \frac{25}{2} \end{aligned}$$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b)} \quad \int_1^6 g(x) \, dx &= \int_1^3 g(x) \, dx + \int_3^6 g(x) \, dx \\ &= \int_1^3 2 \, dx + \int_3^6 2(x-4)^2 \, dx \\ &= 4 + \left[\frac{2}{3}(x-4)^3 \right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3} \right) = 10 \end{aligned}$$

3 : $\begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f''(x) = g'(x)$ is increasing on those intervals.

2 : $\begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.

2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

2018 SCORING GUIDELINES

Question 4

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$.

By the Mean Value Theorem, there exists a value c , $3 < c < 5$, such that $H'(c) = 2$.

$$(c) \quad \text{The average height of the tree over the time interval } 2 \leq t \leq 10 \text{ is given by } \frac{1}{10 - 2} \int_2^{10} H(t) \, dt.$$

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) \, dt &\approx \frac{1}{8} \left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval $2 \leq t \leq 10$ is $\frac{263}{32}$ meters.

$$(d) \quad G(x) = 50 \Rightarrow x = 1$$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

$$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$$

$$2 : \begin{cases} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$$

$$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$$

$$3 : \begin{cases} 2 : \frac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$$

Note: max 1/3 [1-0] if no chain rule

2018 SCORING GUIDELINES

Question 5

$$(a) \text{ Area} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} (4^2 - (3 + 2\cos\theta)^2) d\theta$$

$$(b) \frac{dr}{d\theta} = -2\sin\theta \Rightarrow \left. \frac{dr}{d\theta} \right|_{\theta=\pi/2} = -2$$

$$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$$

$$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \left. \frac{dy/d\theta}{dx/d\theta} \right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$

at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

— OR —

$$y = r\sin\theta = (3 + 2\cos\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$$

$$x = r\cos\theta = (3 + 2\cos\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \left. \frac{dy/d\theta}{dx/d\theta} \right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$

at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

$$(c) \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = -2\sin\theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin\theta}$$

$$\left. \frac{d\theta}{dt} \right|_{\theta=\pi/3} = 3 \cdot \frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \\ \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ 1 : \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ 1 : \text{answer} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} \\ 1 : \frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin\theta} \\ 1 : \text{answer with units} \end{cases}$$

2018 SCORING GUIDELINES

Question 6

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

(b)
$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$\left| \frac{x}{3} \right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x \ln\left(1 + \frac{x}{3}\right)$ converges absolutely for $\left|\frac{x}{3}\right| < 1$.

$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

5 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

— OR —

5 : $\begin{cases} 1 : \text{radius for } \ln(1+x) \text{ series} \\ 1 : \text{substitutes } \frac{x}{3} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

2 : $\begin{cases} 1 : \text{uses fifth-degree term as error bound} \\ 1 : \text{answer} \end{cases}$

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AP Calculus BC

Scoring Guidelines

2017 SCORING GUIDELINES

Question 1

(a) Volume = $\int_0^{10} A(h) \, dh$ $\approx (2 - 0) \cdot A(0) + (5 - 2) \cdot A(2) + (10 - 5) \cdot A(5)$ $= 2 \cdot 50.3 + 3 \cdot 14.4 + 5 \cdot 6.5$ $= 176.3$ cubic feet (b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing. (c) $\int_0^{10} f(h) \, dh = 101.325338$ The volume is 101.325 cubic feet. (d) Using the model, $V(h) = \int_0^h f(x) \, dx$. $\begin{aligned} \left. \frac{dV}{dt} \right _{h=5} &= \left[\frac{dV}{dh} \cdot \frac{dh}{dt} \right]_{h=5} \\ &= \left[f(h) \cdot \frac{dh}{dt} \right]_{h=5} \\ &= f(5) \cdot 0.26 = 1.694419 \end{aligned}$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	1 : units in parts (a), (c), and (d) 2 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$ 1 : overestimate with reason 2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ 3 : $\begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$
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2017 SCORING GUIDELINES

Question 2

$$(a) \frac{1}{2} \int_0^{\pi/2} (f(\theta))^2 d\theta = 0.648414$$

The area of R is 0.648.

$$(b) \int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \frac{1}{2} \int_0^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$$

— OR —

$$\int_0^k ((g(\theta))^2 - (f(\theta))^2) d\theta = \int_k^{\pi/2} ((g(\theta))^2 - (f(\theta))^2) d\theta$$

$$(c) w(\theta) = g(\theta) - f(\theta)$$

$$w_A = \frac{\int_0^{\pi/2} w(\theta) d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$$(d) w(\theta) = w_A \text{ for } 0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \theta = 0.517688$$

$$w(\theta) = w_A \text{ at } \theta = 0.518 \text{ (or 0.517).}$$

$$w'(0.518) < 0 \Rightarrow w(\theta) \text{ is decreasing at } \theta = 0.518.$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{integral expression} \\ \quad \text{for one region} \\ 1 : \text{equation} \end{cases}$$

$$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$$

$$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$$

2017 SCORING GUIDELINES

Question 3

$$(a) \quad f(-6) = f(-2) + \int_{-2}^{-6} f'(x) \, dx = 7 - \int_{-6}^{-2} f'(x) \, dx = 7 - 4 = 3$$

$$f(5) = f(-2) + \int_{-2}^5 f'(x) \, dx = 7 - 2\pi + 3 = 10 - 2\pi$$

$$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$$

$$(b) \quad f'(x) > 0 \text{ on the intervals } [-6, -2) \text{ and } (2, 5).$$

Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

$$2 : \text{answer with justification}$$

$$(c) \quad \text{The absolute minimum will occur at a critical point where } f'(x) = 0 \text{ or at an endpoint.}$$

$$f'(x) = 0 \Rightarrow x = -2, x = 2$$

$$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$$(d) \quad f''(-5) = \frac{2 - 0}{-6 - (-2)} = -\frac{1}{2}$$

$$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad \text{with explanation} \end{cases}$$

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2 \text{ and } \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1$$

$f''(3)$ does not exist because

$$\lim_{x \rightarrow 3^-} \frac{f'(x) - f'(3)}{x - 3} \neq \lim_{x \rightarrow 3^+} \frac{f'(x) - f'(3)}{x - 3}.$$

2017 SCORING GUIDELINES

Question 4

$$(a) \quad H'(0) = -\frac{1}{4}(91 - 27) = -16$$

$$H(0) = 91$$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$(b) \quad \frac{d^2H}{dt^2} = -\frac{1}{4} \frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$$

$$H > 27 \text{ for } t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0 \text{ for } t > 0$$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

$$(c) \quad \frac{dG}{(G - 27)^{2/3}} = -dt$$

$$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1) dt$$

$$3(G - 27)^{1/3} = -t + C$$

$$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$$

$$3(G - 27)^{1/3} = 12 - t$$

$$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3 \text{ for } 0 \leq t < 10$$

The internal temperature of the potato at time $t = 3$ minutes is

$$27 + \left(\frac{12 - 3}{3}\right)^3 = 54 \text{ degrees Celsius.}$$

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

1 : underestimate with reason

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

2017 SCORING GUIDELINES

Question 5

$$(a) f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2}$$

$$f'(3) = \frac{(-3)(5)}{(18 - 21 + 5)^2} = -\frac{15}{4}$$

$$(b) f'(x) = \frac{-3(4x - 7)}{(2x^2 - 7x + 5)^2} = 0 \Rightarrow x = \frac{7}{4}$$

The only critical point in the interval $1 < x < 2.5$ has x -coordinate $\frac{7}{4}$.

f' changes sign from positive to negative at $x = \frac{7}{4}$.

Therefore, f has a relative maximum at $x = \frac{7}{4}$.

$$\begin{aligned} (c) \int_5^{\infty} f(x) dx &= \lim_{b \rightarrow \infty} \int_5^b \frac{3}{2x^2 - 7x + 5} dx = \lim_{b \rightarrow \infty} \int_5^b \left(\frac{2}{2x - 5} - \frac{1}{x - 1} \right) dx \\ &= \lim_{b \rightarrow \infty} \left[\ln(2x - 5) - \ln(x - 1) \right]_5^b = \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2x - 5}{x - 1} \right) \right]_5^b \\ &= \lim_{b \rightarrow \infty} \left[\ln \left(\frac{2b - 5}{b - 1} \right) - \ln \left(\frac{5}{4} \right) \right] = \ln 2 - \ln \left(\frac{5}{4} \right) = \ln \left(\frac{8}{5} \right) \end{aligned}$$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\int_5^{\infty} \frac{3}{2x^2 - 7x + 5} dx$ converges.

— OR —

$$\frac{3}{2n^2 - 7n + 5} > 0 \text{ and } \frac{1}{n^2} > 0 \text{ for } n \geq 5.$$

Since $\lim_{n \rightarrow \infty} \frac{\frac{3}{2n^2 - 7n + 5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\sum_{n=5}^{\infty} \frac{1}{n^2}$ converges,

the series $\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges by the limit comparison test.

2 : $f'(3)$

2 : $\begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \text{with justification} \end{cases}$

3 : $\begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

2 : answer with conditions

2017 SCORING GUIDELINES

Question 6

(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\frac{(-1)^{n+1}x^n}{n}$.

(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

$$\begin{aligned} \text{(c)} \quad \int_0^x f(t) dt &= \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt \\ &= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x} \\ &= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots \end{aligned}$$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded

by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}.$

$$3 : \begin{cases} 1 : f''(0), f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$$

2 : converges conditionally
with reason

$$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$$

1 : error bound

AP[®] Calculus BC

2016 Scoring Guidelines

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2016 SCORING GUIDELINES

Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \leq t \leq 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

- (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.
- (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- (d) For $0 \leq t \leq 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a) $R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120$ liters/hr²

2 : $\begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t) dt$.

$$\begin{aligned} \int_0^8 R(t) dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

3 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $\text{Total} \approx 50000 + \int_0^8 W(t) dt - 8050$
 $= 50000 + 7836.195325 - 8050 \approx 49786$ liters

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

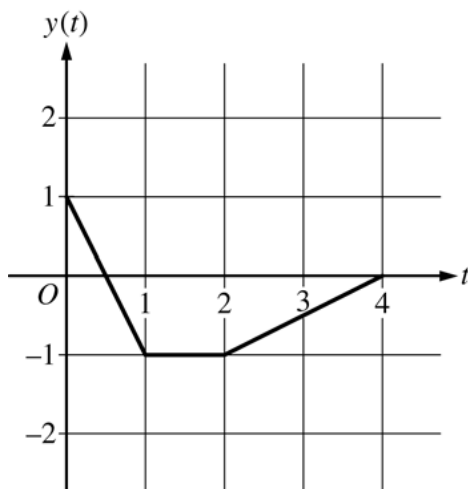
2 : $\begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

Therefore, the Intermediate Value Theorem guarantees at least one time t , $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t , the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

2016 SCORING GUIDELINES

Question 2



At time t , the position of a particle moving in the xy -plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y , consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

- Find the position of the particle at $t = 3$.
- Find the slope of the line tangent to the path of the particle at $t = 3$.
- Find the speed of the particle at $t = 3$.
- Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$$(a) \quad x(3) = x(0) + \int_0^3 x'(t) dt = 5 + 9.377035 = 14.377$$

$$y(3) = -\frac{1}{2}$$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

$$(b) \quad \text{Slope} = \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$$

$$(c) \quad \text{Speed} = \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969 \text{ (or } 9.968)$$

$$\begin{aligned} (d) \quad \text{Distance} &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2} dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2} dt \\ &= 2.237871 + 2.112003 = 4.350 \text{ (or } 4.349) \end{aligned}$$

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

1 : slope

2 : $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

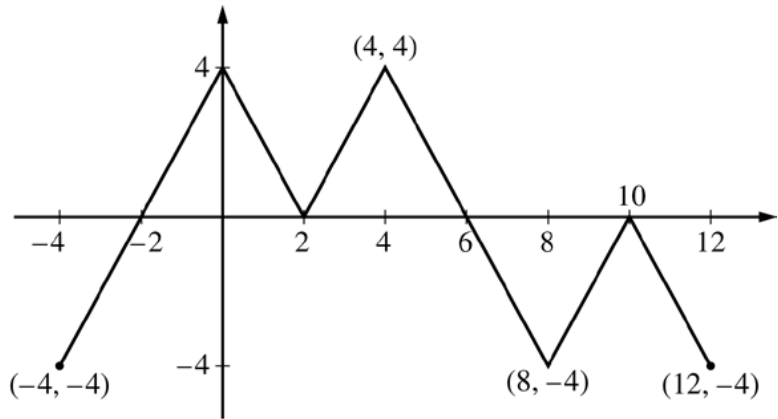
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Question 3

The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by

$$g(x) = \int_2^x f(t) \, dt.$$

- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

Graph of f

- (a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \leq 0$ for $8 \leq x \leq 12$.
- (b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 8$.
- (c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

- (d) $g(x) \leq 0$ for $-4 \leq x \leq 2$ and $10 \leq x \leq 12$.

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 : $\begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \quad \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$

2 : intervals

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Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y .

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$
 Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$$(a) \quad \frac{d^2y}{dx^2} = 2x - \frac{1}{2} \frac{dy}{dx} = 2x - \frac{1}{2} \left(x^2 - \frac{1}{2}y \right)$$

$$2 : \frac{d^2y}{dx^2} \text{ in terms of } x \text{ and } y$$

$$(b) \quad \left. \frac{dy}{dx} \right|_{(x,y)=(-2,8)} = (-2)^2 - \frac{1}{2} \cdot 8 = 0$$

2 : conclusion with justification

$$\left. \frac{d^2y}{dx^2} \right|_{(x,y)=(-2,8)} = 2(-2) - \frac{1}{2} \left((-2)^2 - \frac{1}{2} \cdot 8 \right) = -4 < 0$$

Thus, the graph of f has a relative maximum at the point $(-2, 8)$.

$$(c) \quad \lim_{x \rightarrow -1} (g(x) - 2) = 0 \text{ and } \lim_{x \rightarrow -1} 3(x+1)^2 = 0$$

$$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right) = \lim_{x \rightarrow -1} \left(\frac{g'(x)}{6(x+1)} \right)$$

$$\lim_{x \rightarrow -1} g'(x) = 0 \text{ and } \lim_{x \rightarrow -1} 6(x+1) = 0$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left(\frac{g'(x)}{6(x+1)} \right) = \lim_{x \rightarrow -1} \left(\frac{g''(x)}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$$

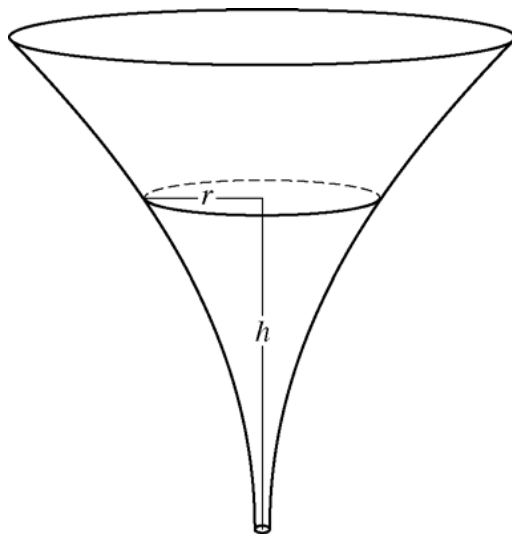
$$(d) \quad h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1) \cdot \frac{1}{2} = \frac{3}{2}$$

$$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{5}{4}$$

$$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$$

2016 SCORING GUIDELINES

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h , the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \leq h \leq 10$. The units of r and h are inches.

- (a) Find the average value of the radius of the funnel.
- (b) Find the volume of the funnel.
- (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\begin{aligned} \text{(a) Average radius} &= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[3h + \frac{h^3}{3} \right]_0^{10} \\ &= \frac{1}{200} \left(\left(30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60} \text{ in} \end{aligned}$$

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_0^{10} \left(\left(\frac{1}{20} \right) (3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh \\ &= \frac{\pi}{400} \left[9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} \\ &= \frac{\pi}{400} \left(\left(90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40} \text{ in}^3 \end{aligned}$$

3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(c) } \frac{dr}{dt} &= \frac{1}{20}(2h) \frac{dh}{dt} \\ -\frac{1}{5} &= \frac{3}{10} \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec} \end{aligned}$$

3 : $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2016 SCORING GUIDELINES

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence.

It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the n th derivative of f at $x = 1$ is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
- (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.
- (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1$, $f'(1) = -\frac{1}{2}$, $f''(1) = \frac{1}{2^2}$, $f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \dots$$

$$+ \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \dots$$

$$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$$

- (b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$.

Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \dots$.

Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

$$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

- (d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$$

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$