

Part B (AB or BC): Graphing calculator not allowed**Question 3****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R , where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

	Model Solution	Scoring	
A	Approximate $R'(1)$ using the average rate of change of R over the interval $0 \leq t \leq 2$. Show the work that leads to your answer. Indicate units of measure.		
	$R'(1) \approx \frac{R(2) - R(0)}{2 - 0}$ $= \frac{100 - 90}{2} = \frac{10}{2} = 5 \text{ words per minute per minute}$	Answer with setup	Point 1 (P1)
		Units	Point 2 (P2)

Scoring Notes for Part A

- To earn **P1**, a response must present the answer along with the supporting work of a difference and a quotient using values from the table.
 - $\frac{100 - 90}{2 - 0}$, $\frac{10}{2 - 0}$, $\frac{100 - 90}{2}$, or $\frac{R(2) - R(0)}{2 - 0} = 5$ is sufficient to earn **P1**.
 - $\frac{R(2) - R(0)}{2 - 0}$ by itself is not sufficient to earn **P1**.
- P2** is earned for correct units, whether or not they are attached to a numerical value for the average rate of change.
- P2** is also earned for the units “words / minute² .”

B Must there be a value c , for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

R is differentiable implies R is continuous.

Differentiable
implies continuous

Point 3 (P3)

$$R(0) = 90 < 155 < R(10) = 162$$

Answer with
justification

Point 4 (P4)

Therefore, by the Intermediate Value Theorem, there must be a value c , with $0 < c < 10$, such that $R(c) = 155$.

Scoring Notes for Part B

- To earn **P3**, a response must state that R is continuous because R is differentiable (or equivalent). A response that simply states “ R is continuous” without justification does not earn **P3**.
- A response does not need to earn **P3** to be eligible for **P4**.
- To earn **P4**, a response must indicate that $R(0) < 155$ (or $R(2) < 155$ or $R(8) < 155$) and $R(10) > 155$, state that “ R is continuous,” and answer “yes” in some way.
- To earn **P4**, a response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

- C Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t) dt$. Show the work that leads to your answer.

$\int_0^{10} R(t) dt$ $\approx \frac{R(0) + R(2)}{2}(2 - 0) + \frac{R(2) + R(8)}{2}(8 - 2)$ $+ \frac{R(8) + R(10)}{2}(10 - 8)$	Form of trapezoidal sum	Point 5 (P5)
$= \frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ $= \frac{190}{2}(2) + \frac{250}{2}(6) + \frac{312}{2}(2) = 190 + 750 + 312 = 1252$	Answer with supporting work	Point 6 (P6)

Scoring Notes for Part C

- Read “=” as “ $\approx$ ” for **P5**.
- The form of a trapezoidal sum includes three terms, each of which includes a product of two factors, where one of the factors incorporates the $\frac{1}{2}$ as part of the product. To earn **P5**, at least five of the six factors must be correct. If any of the six factors is incorrect, the response does not earn **P6**. Consider the following examples:
 - $\frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ earns **P5** and is sufficient to earn **P6**.
 - $\frac{190}{2}(2) + \frac{250}{2}(6) + \frac{312}{2}(2)$ earns **P5** and is sufficient to earn **P6**.
 - $\frac{1}{2}((R(0) + R(2))(2) + (R(2) + R(8))(6) + (R(8) + R(10))(2))$ earns **P5** and is eligible for **P6**.
 - $\frac{90 + 100}{2}(2) + \frac{100 + 150}{2}(2) + \frac{150 + 162}{2}(2)$ earns **P5** but is not eligible for **P6**.
(Note that the factor of 2 in the second term of this expression is incorrect.)
- Special case:** A response of $(90 + 100) + (100 + 150)3 + (150 + 162)$ earns both **P5** and **P6**.
- To be eligible for **P6**, a response must have earned **P5**.
Special case: A response of $95 \cdot 2 + 125 \cdot 6 + 156 \cdot 2$ earns **P6** but does not earn **P5**.
- A response of $\frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ or equivalent earns **P6** (i.e., subsequent errors in simplification will not be considered in scoring for **P6**).
- A response of $\frac{(90 \cdot 2 + 100 \cdot 6 + 150 \cdot 2) + (100 \cdot 2 + 150 \cdot 6 + 162 \cdot 2)}{2}$ or equivalent earns both **P5** and **P6**. (Note that the average of the left Riemann sum and right Riemann sum is equivalent to the trapezoidal sum.)
- A completely correct left Riemann sum (e.g., $90 \cdot 2 + 100 \cdot 6 + 150 \cdot 2 = 1080$) or a completely correct right Riemann sum (e.g., $100 \cdot 2 + 150 \cdot 6 + 162 \cdot 2 = 1424$) earns **P5** but does not earn **P6**.

- D** A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

$$\int_0^{10} W(t) dt = \int_0^{10} \left(-\frac{3}{10}t^2 + 8t + 100 \right) dt$$

Integrand **Point 7 (P7)**

$$= \left(-\frac{1}{10}t^3 + 4t^2 + 100t \right) \Big|_0^{10}$$

Antiderivative **Point 8 (P8)**

$$= \left(-\frac{1}{10} \cdot 1000 + 4 \cdot 100 + 100 \cdot 10 \right) - \left(-\frac{1}{10} \cdot 0 + 4 \cdot 0 + 100 \cdot 0 \right) \\ = 1300$$

Answer **Point 9 (P9)**

Based on the model, the teacher has read 1300 words by the end of the 10 minutes.

Scoring Notes for Part D

- **P7** is earned for an indefinite or definite integral with integrand $W(t)$, with or without the differential dt .
- **P8** is earned for the correct antiderivative, with or without the constant of integration.
- To be eligible for **P9**, a response must have earned **P8**.
- A response of $\left(-\frac{1}{10} \cdot 1000 + 4 \cdot 100 + 100 \cdot 10 \right) - \left(-\frac{1}{10} \cdot 0 + 4 \cdot 0 + 100 \cdot 0 \right)$ or equivalent banks **P9** (i.e., subsequent errors in simplification will not be considered in scoring for **P9**).

Part B (AB or BC): Graphing calculator not allowed**Question 4****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r , where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \leq t \leq 12$.

Model Solution**Scoring**

- (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \leq t \leq 10$. Show the computations that lead to your answer, and indicate units of measure.

$$r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (-4.4)}{10 - 7}$$

$r''(8.5)$ with
supporting work **1 point**

$$= \frac{0.6}{3} = 0.2 \text{ centimeter per day per day}$$

Units **1 point**

Scoring notes:

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as cm/day^2).

Total for part (a) 2 points

(b) Is there a time t , $0 \leq t \leq 3$, for which $r'(t) = -6$? Justify your answer.

$r(t)$ is twice-differentiable. $\Rightarrow r'(t)$ is differentiable. $\Rightarrow r'(t)$ is continuous.	$r'(0) < -6 < r'(3)$	1 point
$r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time t , $0 < t < 3$, such that $r'(t) = -6$.	Conclusion using Intermediate Value Theorem	1 point

Scoring notes:

- To earn the first point, the response must establish that -6 is between $r'(0)$ and $r'(3)$ (or -6.1 and -5). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of “ $r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$ ” does not state that -6 is between -6.1 and -5 . Thus this response does not earn the first point.
- To earn the second point:
 - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
 - The response must have earned the first point.
 - Exception: A response of “ $r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$ ” does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
 - The response must conclude that there is a time t such that $r'(t) = -6$. (A statement of “yes” would be sufficient.)
- To earn the second point, the response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

Total for part (b) 2 points

- (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t) dt$.

$\int_0^{12} r'(t) dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$	Form of right Riemann sum	1 point
$= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$	Answer	1 point

Scoring notes:

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15, -17.6, -11.4, -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer (-51) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t) dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$) earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t) dt$ earns no points.
- Units are not required or read in this part.

Total for part (c) 2 points

- (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$	Product rule	1 point
	Chain rule	1 point
$\left. \frac{dV}{dt} \right _{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70,000\pi}{3}$	Answer	1 point
The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70,000\pi}{3}$ cubic centimeters per day.		

Scoring notes:

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi r h + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats r or h (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$ is correct if h is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} \frac{dh}{dt}$ earns any points.
- A response that assumes a functional relationship between r and h (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \rightarrow V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \rightarrow \frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

Total for part (d) 3 points

Total for question 4 9 points

AP[®] Calculus BC

2016 Scoring Guidelines

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Question 1

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \leq t \leq 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

- (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.
- (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- (d) For $0 \leq t \leq 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

(a) $R'(2) \approx \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{3 - 1} = -120$ liters/hr²

2 : $\begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t) dt$.

$$\begin{aligned} \int_0^8 R(t) dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters} \end{aligned}$$

This is an overestimate since R is a decreasing function.

3 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $\text{Total} \approx 50000 + \int_0^8 W(t) dt - 8050$
 $= 50000 + 7836.195325 - 8050 \approx 49786$ liters

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

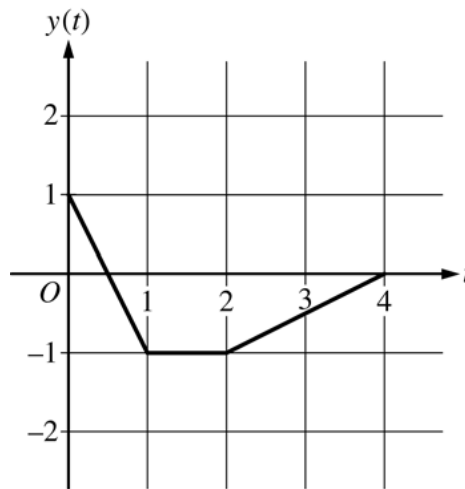
2 : $\begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

Therefore, the Intermediate Value Theorem guarantees at least one time t , $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of t , the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

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Question 2



At time t , the position of a particle moving in the xy -plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y , consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

- Find the position of the particle at $t = 3$.
- Find the slope of the line tangent to the path of the particle at $t = 3$.
- Find the speed of the particle at $t = 3$.
- Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$$(a) \quad x(3) = x(0) + \int_0^3 x'(t) dt = 5 + 9.377035 = 14.377$$

$$y(3) = -\frac{1}{2}$$

The position of the particle at $t = 3$ is $(14.377, -0.5)$.

$$(b) \quad \text{Slope} = \frac{y'(3)}{x'(3)} = \frac{0.5}{9.956376} = 0.05$$

$$(c) \quad \text{Speed} = \sqrt{(x'(3))^2 + (y'(3))^2} = 9.969 \text{ (or } 9.968)$$

$$\begin{aligned} (d) \quad \text{Distance} &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= \int_0^1 \sqrt{(x'(t))^2 + (-2)^2} dt + \int_1^2 \sqrt{(x'(t))^2 + 0^2} dt \\ &= 2.237871 + 2.112003 = 4.350 \text{ (or } 4.349) \end{aligned}$$

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

1 : slope

2 : $\begin{cases} 1 : \text{expression for speed} \\ 1 : \text{answer} \end{cases}$

3 : $\begin{cases} 1 : \text{expression for distance} \\ 1 : \text{integrals} \\ 1 : \text{answer} \end{cases}$

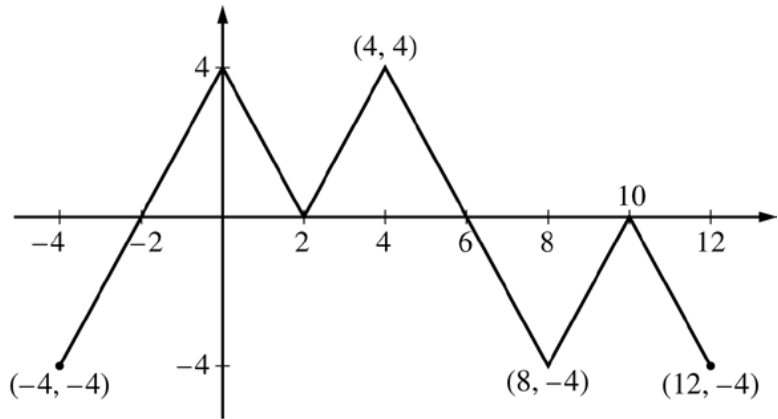
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Question 3

The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by

$$g(x) = \int_2^x f(t) \, dt.$$

- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

Graph of f

- (a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \leq 0$ for $8 \leq x \leq 12$.
- (b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \leq x \leq 4$ and decreasing for $4 \leq x \leq 8$.
- (c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \leq x \leq 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

- (d) $g(x) \leq 0$ for $-4 \leq x \leq 2$ and $10 \leq x \leq 12$.

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

1 : answer with justification

1 : answer with justification

4 : $\left\{ \begin{array}{l} 1 : \text{considers } x = -2 \text{ and } x = 6 \\ \quad \text{as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{array} \right.$

2 : intervals

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Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y .

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$$\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right).$$
 Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$$(a) \quad \frac{d^2y}{dx^2} = 2x - \frac{1}{2} \frac{dy}{dx} = 2x - \frac{1}{2} \left(x^2 - \frac{1}{2}y \right)$$

$$2 : \frac{d^2y}{dx^2} \text{ in terms of } x \text{ and } y$$

$$(b) \quad \left. \frac{dy}{dx} \right|_{(x,y)=(-2,8)} = (-2)^2 - \frac{1}{2} \cdot 8 = 0$$

2 : conclusion with justification

$$\left. \frac{d^2y}{dx^2} \right|_{(x,y)=(-2,8)} = 2(-2) - \frac{1}{2} \left((-2)^2 - \frac{1}{2} \cdot 8 \right) = -4 < 0$$

Thus, the graph of f has a relative maximum at the point $(-2, 8)$.

$$(c) \quad \lim_{x \rightarrow -1} (g(x) - 2) = 0 \text{ and } \lim_{x \rightarrow -1} 3(x+1)^2 = 0$$

$$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right) = \lim_{x \rightarrow -1} \left(\frac{g'(x)}{6(x+1)} \right)$$

$$\lim_{x \rightarrow -1} g'(x) = 0 \text{ and } \lim_{x \rightarrow -1} 6(x+1) = 0$$

Using L'Hospital's Rule,

$$\lim_{x \rightarrow -1} \left(\frac{g'(x)}{6(x+1)} \right) = \lim_{x \rightarrow -1} \left(\frac{g''(x)}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$$

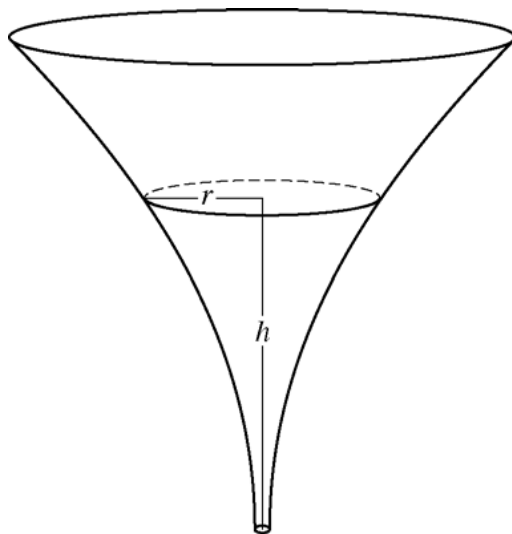
$$(d) \quad h\left(\frac{1}{2}\right) \approx h(0) + h'(0) \cdot \frac{1}{2} = 2 + (-1) \cdot \frac{1}{2} = \frac{3}{2}$$

$$h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right) \cdot \frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{5}{4}$$

$$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$$

2016 SCORING GUIDELINES

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h , the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \leq h \leq 10$. The units of r and h are inches.

- (a) Find the average value of the radius of the funnel.
- (b) Find the volume of the funnel.
- (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\begin{aligned} \text{(a) Average radius} &= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[3h + \frac{h^3}{3} \right]_0^{10} \\ &= \frac{1}{200} \left(\left(30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60} \text{ in} \end{aligned}$$

3 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_0^{10} \left(\left(\frac{1}{20} \right) (3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh \\ &= \frac{\pi}{400} \left[9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} \\ &= \frac{\pi}{400} \left(\left(90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40} \text{ in}^3 \end{aligned}$$

3 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(c) } \frac{dr}{dt} &= \frac{1}{20}(2h) \frac{dh}{dt} \\ -\frac{1}{5} &= \frac{3}{10} \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec} \end{aligned}$$

3 : $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2016 SCORING GUIDELINES

Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence.

It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the n th derivative of f at $x = 1$ is given by

$$f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n} \text{ for } n \geq 2.$$

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
- (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.
- (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.
- (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$$\begin{aligned} \text{(a)} \quad f(1) = 1, \quad f'(1) = -\frac{1}{2}, \quad f''(1) = \frac{1}{2^2}, \quad f'''(1) = -\frac{2}{2^3} \\ f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots \\ + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots \end{aligned}$$

$$4 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \\ 1 : \text{general term} \end{cases}$$

- (b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.

Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

$$2 : \begin{cases} 1 : \text{identifies both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

$$\text{(c)} \quad f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$$

1 : approximation

- (d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$2 : \begin{cases} 1 : \text{error form} \\ 1 : \text{analysis} \end{cases}$$

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$