

Part A (AB or BC): Graphing calculator required**Question 1****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

	Model Solution	Scoring	
A	Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.		
	$\frac{1}{4 - 0} \int_0^4 C(t) \, dt$	Average value formula	Point 1 (P1)
	$= \frac{1}{4}(11.112896) = 2.778224$	Answer	Point 2 (P2)
	From time $t = 0$ to $t = 4$ weeks, the average number of acres affected by the invasive species was 2.778 acres.		

Scoring Notes for Part A

- **P1** is earned for the correct integral, with or without the differential, along with evidence of division by 4. In the presence of the correct integral, the correct answer will suffice as evidence of division by 4. These may appear all in one step, as in the model solution, or in multiple steps.
- **P2** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point.
- Incorrect or unclear communication between the correct integral and the correct answer is treated as scratch work and is not considered in scoring. For example:

○ $\int_0^4 C(t) dt = 11.112896$ so the average velocity is 2.778224.

Note: This response earns **P1** for the correct integral with the correct answer as evidence of division by 4. It also earns **P2** for the correct answer.

○ $\int_0^4 C(t) dt = \frac{11.112896}{4} = 2.778224$

Note: This response earns **P1** for the correct integral with the correct answer as evidence of division by 4. It also earns **P2** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)

○ $\int_0^4 C(t) dt = 2.778224$

Note: This response earns **P1** for the correct integral with the correct answer as evidence of division by 4. It also earns **P2** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)

- Note that the values $\frac{1}{4}(11.112)$ and $\frac{1}{4}(11.113)$ are accurate to three digits after the decimal and therefore earn **P2**.

B Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \leq t \leq 4$. Show the setup for your calculations.	
$\frac{C(4) - C(0)}{4 - 0} = 1.282008$	Uses average rate of change Point 3 (P3)
$C'(t) = \frac{38}{25 + t^2} = 1.282008 \Rightarrow t = 2.154298$ The instantaneous rate of change of C equals the average rate of change of C over the interval $0 \leq t \leq 4$ at time $t = 2.154$.	Answer with supporting work Point 4 (P4)

Scoring Notes for Part B

- P3** may be earned by presenting the expression or value for the average rate of change. Note that because $C(0) = 0$ and the interval is $0 \leq t \leq 4$, any of the following will earn **P3**: $\frac{\int_0^4 C'(t) dt}{4}$, $\frac{C(4) - C(0)}{4 - 0}$, $\frac{C(4)}{4}$, $\frac{5.128 - 0}{4 - 0}$, $\frac{5.128}{4}$, or 1.282. However, neither **P3** nor **P4** is earned by just presenting $t = 1.282$.
 - P4** is earned for the correct answer supported by the appropriate equation. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.
- The following response, for example, earns both **P3** and **P4**: $C'(t) = \frac{C(4) - C(0)}{4}$ when $t = 2.154$.

C Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.	
$\lim_{t \rightarrow \infty} C'(t) = \lim_{t \rightarrow \infty} \frac{38}{25 + t^2}$	Limit expression Point 5 (P5)
$= 0$	Value Point 6 (P6)

Scoring Notes for Part C

- P5** can be earned for either $\lim_{t \rightarrow \infty} C'(t)$ or $\lim_{t \rightarrow \infty} C(t)$.
- A response that includes $\lim_{t \rightarrow \infty} C(t)$ is not eligible to earn **P6**.
- For **P6**, arithmetic with infinity, e.g., $\frac{38}{25 + \infty^2} = 0$, will be considered as scratch work and will not be considered in scoring.

- D** At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A , defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x) \, dx$, models the number of acres affected by the species over the time interval $4 \leq t \leq 36$. At what time t , for $4 \leq t \leq 36$, does A attain its maximum value? Justify your answer.

$$A'(t) = C'(t) - 0.1 \cdot \ln t$$

Considers $A'(t) = 0$ **Point 7 (P7)**

For $4 \leq t \leq 36$, the maximum value of $A(t)$ occurs when $A'(t) = 0$ or at an endpoint.

$$A'(t) = C'(t) - 0.1 \cdot \ln t = 0 \Rightarrow C'(t) = 0.1 \cdot \ln t$$

$$\Rightarrow t = 11.441700$$

Justification **Point 8 (P8)**

t	$A(t)$
4	5.128031
11.441700	7.316978
36	1.743056

Therefore, the number of acres affected by the species is a maximum at time $t = 11.442$ (or 11.441) weeks.

Answer with supporting work **Point 9 (P9)**

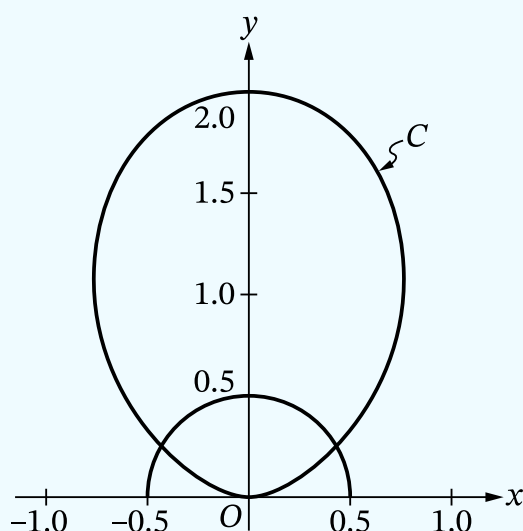
Scoring Notes for Part D

- P7** is earned for considering $A'(t) = 0$, $C'(t) - 0.1 \cdot \ln t = 0$, or $C'(t) = 0.1 \cdot \ln t$. **P7** is not earned by just presenting $t = 11.441700$.
A response that discusses the sign of $A'(t)$ changing or uses the phrase “critical points of A ” also earns **P7**.
- To earn **P8** using a candidates test, a response must make a global argument by correctly evaluating $A(t)$ at $t = 4$, $t = 11.441700$, and $t = 36$. The evaluations must be correct to the first digit after the decimal, rounded or truncated.
- Alternate justifications:
 - $A'(t) > 0$ for $4 < t < 11.442$, and $A'(t) < 0$ for $11.442 < t < 36$. Therefore, $t = 11.442$ is the location of the absolute maximum for A on the interval $4 \leq t \leq 36$.
 - Because $A'(t)$ changes sign from positive to negative at $t = 11.442$ (this might be presented as “ $A'(t) > 0$ for $t < 11.442$, and $A'(t) < 0$ for $t > 11.442$ ”), it is the location of a relative maximum for A . And because $t = 11.442$ is the only critical point of A in the interval $4 \leq t \leq 36$, it is the location of the absolute maximum for A on the interval.
- A response that presents a local argument (such as a First Derivative Test or a Second Derivative Test) or an incorrect global argument does not earn **P8** but is eligible for **P9** with the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

Part A (BC): Graphing calculator required**Question 2****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Curve C is defined by the polar equation $r(\theta) = 2\sin^2 \theta$ for $0 \leq \theta \leq \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \leq \theta \leq \pi$ are shown in the xy -plane.



(Note: Your calculator should be in radian mode.)

	Model Solution	Scoring
A	Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.	
	$\left. \frac{dr}{d\theta} \right _{\theta=1.3} = 1.031003$ The rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$ is 1.031.	Answer with setup Point 1 (P1)

Scoring Notes for Part A

- An exact answer of $\left. \frac{dr}{d\theta} \right|_{\theta=1.3} = 4\sin(1.3)\cos(1.3)$ earns **P1**.
- To earn **P1**, a response must indicate differentiation of r and provide the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point.
 - Examples of responses with correct communication include $\left. \frac{dr}{d\theta} \right|_{\theta=1.3} = 1.031$ and $r'(1.3) = 1.031$.
 - Responses with incorrect communication, such as $r'(\theta) = 1.031$, $r' = 1.031$, or $\frac{dr}{d\theta} = 1.031$, are sufficient to earn **P1**.

B

Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$.

Show the setup for your calculations.

For $0 \leq \theta \leq \pi$, $r(\theta) = \frac{1}{2}$ for $\theta = \theta_1 = \frac{\pi}{6} = 0.523599$ and

$\theta = \theta_2 = \frac{5\pi}{6} = 2.617994$.

$$\frac{1}{2} \int_{\theta_1}^{\theta_2} \left((r(\theta))^2 - \left(\frac{1}{2} \right)^2 \right) d\theta$$

$$= 2.066769$$

The area is 2.067 (or 2.066).

Integrand including $(r(\theta))^2$ **Point 2 (P2)**

Integrand **Point 3 (P3)**

Answer **Point 4 (P4)**

Scoring Notes for Part B

- P2** is earned for a definite integral including $(r(\theta))^2$, such as $\int_a^b \left((r(\theta))^2 - \left(\frac{1}{2} \right)^2 \right) d\theta$ or $\int_a^b (r(\theta))^2 d\theta$, with or without the differential $d\theta$.
- P3** is earned for a definite integral (or integrals) with a correct integrand, such as $\int_a^b \left((r(\theta))^2 - \left(\frac{1}{2} \right)^2 \right) d\theta$ or $\int_a^b (r(\theta))^2 d\theta - \int_c^d \left(\frac{1}{2} \right)^2 d\theta$, with or without the differential $d\theta$.
- The limits $\theta_1 = \frac{\pi}{6} = 0.523599$ and $\theta_2 = \frac{5\pi}{6} = 2.617994$ and the factor $\frac{1}{2}$ are assessed in **P4**, not in **P2** or **P3**.
- P4** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

- Incorrect or unclear communication between the correct integral and the correct answer is treated as scratch work and is not considered in scoring. For example:

$$\circ \int_{\pi/6}^{5\pi/6} \left((r(\theta))^2 - \left(\frac{1}{2} \right)^2 \right) d\theta = 4.133538 \text{ so the area is } 2.067.$$

Note: This response earns **P2** and **P3** for the integral. It also earns **P4** for the correct answer.

$$\circ \int_{\pi/6}^{5\pi/6} \left((r(\theta))^2 - \left(\frac{1}{2} \right)^2 \right) d\theta = 2.067$$

Note: This response earns **P2** and **P3** for the integral. It also earns **P4** for the correct answer. (In this instance, incorrect linkage is not considered in scoring.)

- Special case:** An indefinite integral with a correct integrand does not earn **P2**, earns **P3**, and is eligible to earn **P4** with a correct answer.
- A response of $\int_{\pi/6}^{\pi/2} \left((r(\theta))^2 - \left(\frac{1}{2} \right)^2 \right) d\theta = 2.067$, using the symmetry of the region, earns **P2**, **P3**, and **P4**.

C	It can be shown that $\frac{dx}{d\theta} = 4\sin \theta \cos^2 \theta - 2\sin^3 \theta$ for curve C . For $0 \leq \theta \leq \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y -axis. Justify your answer.									
	For $0 \leq \theta \leq \frac{\pi}{2}$, the curve C is in the first quadrant. Thus, a point on the curve will be farthest away from the y -axis when the x -coordinate attains its maximum value. This will either occur when $\frac{dx}{d\theta} = 0$ or at an endpoint of the interval $0 \leq \theta \leq \frac{\pi}{2}$. $\frac{dx}{d\theta} = 0$	Considers $\frac{dx}{d\theta} = 0$ Point 5 (P5)								
	$\Rightarrow \theta = 0.955317$ <table><tr><td>θ</td><td>$x(\theta) = r(\theta)\cos \theta$</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.955317</td><td>0.769800</td></tr><tr><td>$\frac{\pi}{2}$</td><td>0</td></tr></table>	θ	$x(\theta) = r(\theta)\cos \theta$	0	0	0.955317	0.769800	$\frac{\pi}{2}$	0	Justification Point 6 (P6)
θ	$x(\theta) = r(\theta)\cos \theta$									
0	0									
0.955317	0.769800									
$\frac{\pi}{2}$	0									
	Therefore, the value of θ for which the point on the curve is farthest from the y -axis is 0.955.	Answer with supporting work Point 7 (P7)								

Scoring Notes for Part C

- **P5** is earned for considering $\frac{dx}{d\theta} = 0$. **P5** is not earned by just presenting $\theta = 0.955317$.

A response that discusses the sign of $\frac{dx}{d\theta}$ changing or uses the phrase “critical points of $x(\theta)$ ” also earns **P5**.

- The value $\theta = 0.955317$ might be presented as $\arccos\left(\frac{1}{\sqrt{3}}\right)$, $\arcsin\left(\sqrt{\frac{2}{3}}\right)$, or $\arctan(\sqrt{2})$.
- To earn **P6** using a candidates test, a response must make a global argument by correctly evaluating $x(\theta)$ at $\theta = 0$, $\theta = 0.955317$, and $\theta = \frac{\pi}{2}$. The evaluations must be correct to the first digit after the decimal, rounded or truncated.
- Alternate justifications:
 - $\frac{dx}{d\theta} > 0$ for $0 < \theta < 0.955$, and $\frac{dx}{d\theta} < 0$ for $0.955 < \theta < \frac{\pi}{2}$. Therefore, $\theta = 0.955$ is the location of the absolute maximum for $x(\theta)$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$.
 - Because $\frac{dx}{d\theta}$ changes sign from positive to negative at $\theta = 0.955$ (this might be presented as “ $\frac{dx}{d\theta} > 0$ for $\theta < 0.955$, and $\frac{dx}{d\theta} < 0$ for $\theta > 0.955$ ”), it is the location of a relative maximum for $x(\theta)$. And because $\theta = 0.955$ is the only critical point of $x(\theta)$ in the interval $0 \leq \theta \leq \frac{\pi}{2}$, it is the location of the absolute maximum for $x(\theta)$ on the interval.
 - Because $\frac{dx}{d\theta}\bigg|_{\theta=0.955} = 0$ and $\frac{d^2x}{d\theta^2}\bigg|_{\theta=0.955} < 0$, $\theta = 0.955$ is the location of a relative maximum for $x(\theta)$. And because $\theta = 0.955$ is the only critical point of $x(\theta)$ in the interval $0 \leq \theta \leq \frac{\pi}{2}$, it is the location of the absolute maximum for $x(\theta)$ on the interval.
- A response that presents only a local argument (such as a First Derivative Test or a Second Derivative Test) or an incorrect global argument does not earn **P6** but is eligible for **P7** with the correct answer. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

D

A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t . Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

$$\left. \frac{dr}{dt} \right|_{\theta=1.3} = \left(\frac{dr}{d\theta} \cdot \frac{d\theta}{dt} \right) \bigg|_{\theta=1.3} = 1.031003 \cdot 15 = 15.465041$$

$$\frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$$

Point 8 (P8)

Answer

Point 9 (P9)

The particle's distance from the origin changes at a rate of 15.465.

Scoring Notes for Part D

- To earn **P8**, $\frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$ can be presented either symbolically or numerically.
- P8** might be earned in one or more steps.
- A response of $1.031 \cdot 15$ or [answer from part A] $\cdot 15$ earns both **P8** and **P9**.
- P9** is earned for the correct answer, with or without supporting work. A reported answer should be accurate to three places after the decimal point, rounded or truncated. An inappropriately rounded answer does not earn the point, unless an earlier point was not earned due to inappropriate rounding.

Part B (AB or BC): Graphing calculator not allowed**Question 3****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R , where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

	Model Solution	Scoring	
A	Approximate $R'(1)$ using the average rate of change of R over the interval $0 \leq t \leq 2$. Show the work that leads to your answer. Indicate units of measure.		
	$R'(1) \approx \frac{R(2) - R(0)}{2 - 0}$ $= \frac{100 - 90}{2} = \frac{10}{2} = 5 \text{ words per minute per minute}$	Answer with setup Units	Point 1 (P1) Point 2 (P2)

Scoring Notes for Part A

- To earn **P1**, a response must present the answer along with the supporting work of a difference and a quotient using values from the table.
 - $\frac{100 - 90}{2 - 0}$, $\frac{10}{2 - 0}$, $\frac{100 - 90}{2}$, or $\frac{R(2) - R(0)}{2 - 0} = 5$ is sufficient to earn **P1**.
 - $\frac{R(2) - R(0)}{2 - 0}$ by itself is not sufficient to earn **P1**.
- P2** is earned for correct units, whether or not they are attached to a numerical value for the average rate of change.
- P2** is also earned for the units “words / minute² .”

B Must there be a value c , for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

R is differentiable implies R is continuous.

Differentiable
implies continuous

Point 3 (P3)

$$R(0) = 90 < 155 < R(10) = 162$$

Answer with
justification

Point 4 (P4)

Therefore, by the Intermediate Value Theorem, there must be a value c , with $0 < c < 10$, such that $R(c) = 155$.

Scoring Notes for Part B

- To earn **P3**, a response must state that R is continuous because R is differentiable (or equivalent). A response that simply states “ R is continuous” without justification does not earn **P3**.
- A response does not need to earn **P3** to be eligible for **P4**.
- To earn **P4**, a response must indicate that $R(0) < 155$ (or $R(2) < 155$ or $R(8) < 155$) and $R(10) > 155$, state that “ R is continuous,” and answer “yes” in some way.
- To earn **P4**, a response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

- C Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t) dt$. Show the work that leads to your answer.

$\int_0^{10} R(t) dt$ $\approx \frac{R(0) + R(2)}{2}(2 - 0) + \frac{R(2) + R(8)}{2}(8 - 2)$ $+ \frac{R(8) + R(10)}{2}(10 - 8)$	Form of trapezoidal sum	Point 5 (P5)
$= \frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ $= \frac{190}{2}(2) + \frac{250}{2}(6) + \frac{312}{2}(2) = 190 + 750 + 312 = 1252$	Answer with supporting work	Point 6 (P6)

Scoring Notes for Part C

- Read “=” as “ $\approx$ ” for **P5**.
- The form of a trapezoidal sum includes three terms, each of which includes a product of two factors, where one of the factors incorporates the $\frac{1}{2}$ as part of the product. To earn **P5**, at least five of the six factors must be correct. If any of the six factors is incorrect, the response does not earn **P6**. Consider the following examples:
 - $\frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ earns **P5** and is sufficient to earn **P6**.
 - $\frac{190}{2}(2) + \frac{250}{2}(6) + \frac{312}{2}(2)$ earns **P5** and is sufficient to earn **P6**.
 - $\frac{1}{2}((R(0) + R(2))(2) + (R(2) + R(8))(6) + (R(8) + R(10))(2))$ earns **P5** and is eligible for **P6**.
 - $\frac{90 + 100}{2}(2) + \frac{100 + 150}{2}(2) + \frac{150 + 162}{2}(2)$ earns **P5** but is not eligible for **P6**.
(Note that the factor of 2 in the second term of this expression is incorrect.)
- Special case:** A response of $(90 + 100) + (100 + 150)3 + (150 + 162)$ earns both **P5** and **P6**.
- To be eligible for **P6**, a response must have earned **P5**.
Special case: A response of $95 \cdot 2 + 125 \cdot 6 + 156 \cdot 2$ earns **P6** but does not earn **P5**.
- A response of $\frac{90 + 100}{2}(2 - 0) + \frac{100 + 150}{2}(8 - 2) + \frac{150 + 162}{2}(10 - 8)$ or equivalent earns **P6** (i.e., subsequent errors in simplification will not be considered in scoring for **P6**).
- A response of $\frac{(90 \cdot 2 + 100 \cdot 6 + 150 \cdot 2) + (100 \cdot 2 + 150 \cdot 6 + 162 \cdot 2)}{2}$ or equivalent earns both **P5** and **P6**. (Note that the average of the left Riemann sum and right Riemann sum is equivalent to the trapezoidal sum.)
- A completely correct left Riemann sum (e.g., $90 \cdot 2 + 100 \cdot 6 + 150 \cdot 2 = 1080$) or a completely correct right Riemann sum (e.g., $100 \cdot 2 + 150 \cdot 6 + 162 \cdot 2 = 1424$) earns **P5** but does not earn **P6**.

- D** A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

$$\int_0^{10} W(t) \, dt = \int_0^{10} \left(-\frac{3}{10}t^2 + 8t + 100 \right) dt$$

Integrand **Point 7 (P7)**

$$= \left(-\frac{1}{10}t^3 + 4t^2 + 100t \right) \Big|_0^{10}$$

Antiderivative **Point 8 (P8)**

$$= \left(-\frac{1}{10} \cdot 1000 + 4 \cdot 100 + 100 \cdot 10 \right) - \left(-\frac{1}{10} \cdot 0 + 4 \cdot 0 + 100 \cdot 0 \right) \\ = 1300$$

Answer **Point 9 (P9)**

Based on the model, the teacher has read 1300 words by the end of the 10 minutes.

Scoring Notes for Part D

- **P7** is earned for an indefinite or definite integral with integrand $W(t)$, with or without the differential dt .
- **P8** is earned for the correct antiderivative, with or without the constant of integration.
- To be eligible for **P9**, a response must have earned **P8**.
- A response of $\left(-\frac{1}{10} \cdot 1000 + 4 \cdot 100 + 100 \cdot 10 \right) - \left(-\frac{1}{10} \cdot 0 + 4 \cdot 0 + 100 \cdot 0 \right)$ or equivalent banks **P9** (i.e., subsequent errors in simplification will not be considered in scoring for **P9**).

Part B (BC): Graphing calculator not allowed**Question 5****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

	Model Solution	Scoring
A	Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.	
	$\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y \frac{dy}{dx}$	Product rule Point 1 (P1)
		Chain rule Point 2 (P2)
	$f'(1) = \left. \frac{dy}{dx} \right _{(x,y)=(1,-1)} = (3 - 1)(-1)^2 = 2$	$f''(1)$ Point 3 (P3)
	$f''(1) = \left. \frac{d^2y}{dx^2} \right _{(x,y)=(1,-1)} = -(-1)^2 + (3 - 1)(2)(-1)(2) = -9$	

Scoring Notes for Part A

- The expression $\frac{d^2y}{dx^2} = -y^2 + (3-x)2y$ or $\frac{d^2y}{dx^2} = 6y - y^2 - 2xy$ earns **P1** but not **P2**. Such a response is not eligible for **P3**.
- The expression $\frac{d^2y}{dx^2} = -2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = 6y\left(\frac{dy}{dx}\right) - 2y\left(\frac{dy}{dx}\right)$ earns **P2** but not **P1**. Such a response is eligible for **P3** for a consistent answer of $f''(1) = 4$ or $f''(1) = -8$, respectively, which is found by correctly substituting correct values for x , y , and $\frac{dy}{dx}$.

- A response of $-(-1)^2 + (3-1)(2)(-1)(2)$ earns **P1**, **P2**, and **P3** regardless of any subsequent errors in simplification.
- Alternate approach (using separation of variables):

The particular solution for the differential equation that passes through the point $(1, -1)$ is

$$y = \frac{2}{x^2 - 6x + 3}. \text{ Therefore, } \frac{dy}{dx} = \frac{-2(2x - 6)}{(x^2 - 6x + 3)^2}.$$

This response has not yet earned **P1**, **P2**, or **P3**.

- A response that correctly applies the quotient rule (or product rule) and the chain rule to find that $\frac{d^2y}{dx^2} = \frac{-4(x^2 - 6x + 3)^2 + 4(2x - 6)(x^2 - 6x + 3)(2x - 6)}{(x^2 - 6x + 3)^4}$ earns **P1** and **P2** and is eligible to earn **P3** for the correct answer of $f''(1) = -9$.
- A response that correctly applies the quotient rule (or product rule) but does not correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4(x^2 - 6x + 3)^2 + 4(2x - 6)(x^2 - 6x + 3)}{(x^2 - 6x + 3)^4}$) earns **P1**, does not earn **P2**, and is not eligible to earn **P3**.
- A response that does not correctly apply the quotient rule (or product rule) but does correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4}{2(x^2 - 6x + 3)(2x - 6)}$) does not earn **P1**, earns **P2**, and is eligible to earn **P3** for a consistent answer.

B Write the second-degree Taylor polynomial for f about $x = 1$.

$$f'(1) = 2 \text{ and } f''(1) = -9$$

Two terms

Point 4 (P4)

$$P_2(x) = -1 + 2(x - 1) - \frac{9}{2}(x - 1)^2$$

Remaining term

Point 5 (P5)

Scoring Notes for Part B

- **P4** and **P5** can be earned with an answer consistent with incorrect values of $f'(1)$ and $f''(1)$ imported from part A.
- Any terms of degree greater than two or “+...” does not earn **P5**.
- A response of $-1 + 2(x - 1) - \frac{9}{2}(x - 1)^2$ earns **P4** and **P5**, regardless of any subsequent algebraic simplification.
- A response that does not present the polynomial as powers of $(x - 1)$ but instead presents a correct expanded/simplified form of the polynomial (e.g., $-\frac{9}{2}x^2 + 11x - \frac{15}{2}$) earns **P4** but not **P5**.

C The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

$$|f(1.1) - P_2(1.1)| \leq \frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3 \leq \frac{60}{6} (0.1)^3 = 0.01$$

Form of error bound

Point 6 (P6)

Analysis

Point 7 (P7)

Scoring Notes for Part C

- **P6** is earned for presenting either $\frac{\max_{1 \leq x \leq 1.1} |f'''(x)|}{3!} |1.1 - 1|^3$ or $\frac{60}{6} (0.1)^3$. Subsequent errors in simplification will not earn **P7**.
- To earn **P7**, a response must have earned **P6** and must explicitly connect the error bound with 0.01; for example by communicating $\text{Error} \leq 0.01$, $\text{Error Bound} = 0.01$, or equivalent.
- A response that declares the error is equal to 0.01 (or any equivalent form of this value) does not earn **P7**.

- D** Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

$$f(1.2) \approx f(1) + (1.2 - 1) \cdot \left. \frac{dy}{dx} \right|_{(1, -1)}$$

$$= -1 + 0.2(2) = -0.6$$

$$f(1.4) \approx f(1.2) + (1.4 - 1.2) \cdot \left. \frac{dy}{dx} \right|_{(1.2, -0.6)}$$

$$\approx -0.6 + 0.2(3 - 1.2)(-0.6)^2 = -0.4704$$

An approximation of $f(1.4)$ is -0.47 .

First step of
Euler's method

Point 8 (P8)

Answer with
supporting work

Point 9 (P9)

Scoring Notes for Part D

- To earn **P8**, a response must demonstrate the first step of Euler's method, with the correct initial condition, correct step size, and correct (or imported) expression for the derivative.

Note: Any subsequent error in simplification or rounding will not affect the scoring for **P8**.

- The two steps of Euler's method may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx} \cdot \Delta x$ (or $\frac{dy}{dx} \cdot 0.2$)
1	-1	0.4
1.2	-0.6	0.1296
1.4	-0.4704	

Note: In the presence of a correct answer, a table does not need to be labeled to earn both **P8** and **P9**. In the presence of no answer or an incorrect answer, such a table must be correctly labeled to earn **P8**.

- A response of $-0.6 + 0.2(3 - 1.2)(-0.6)^2$ earns **P9**, regardless of any subsequent errors in simplification or rounding.
- A response that imports an incorrect value for $f'(1)$ from part A or part B is eligible to earn **P9** with a consistent answer.
- A response may report the final answer as -0.47 , $(1.4, -0.47)$, $-\frac{294}{625}$, or equivalent.

Part B (BC): Graphing calculator not allowed**Question 6****9 points****General Scoring Notes**

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Taylor series for a function f about $x = 4$ is given by

$$\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$$
 and converges to $f(x)$ on its interval of convergence.

	Model Solution	Scoring
A	Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.	
	$\left \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}} \right = \left \frac{(x-4)(n+1)}{3(n+2)} \right $	Sets up ratio Point 1 (P1)
	$\lim_{n \rightarrow \infty} \left \frac{(x-4)(n+1)}{3(n+2)} \right = \left \frac{x-4}{3} \right \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = \left \frac{x-4}{3} \right $	Limit of ratio Point 2 (P2)
	$\left \frac{x-4}{3} \right < 1 \text{ when } -3 < x-4 < 3.$ Thus, the series converges when $1 < x < 7$.	Interior of interval of convergence Point 3 (P3)
	When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3}{n+1}$, which converges by the alternating series test.	Considers both endpoints Point 4 (P4)
	When $x = 7$, the series is $\sum_{n=1}^{\infty} \frac{3}{n+1}$, which diverges by limit comparison to the harmonic series.	Analysis and interval of convergence Point 5 (P5)
	Therefore, the series converges for $1 \leq x < 7$.	

Scoring Notes for Part A

- **P1** is earned by presenting a correct ratio with or without absolute values. Once earned, **P1** is banked (i.e., subsequent errors in simplification or evaluation do not impact scoring for **P1**). **P2** is not earned if there are any errors in simplification or evaluation of the limit.
- **P1** is earned for ratios mathematically equivalent to any of the following:

$$\frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}, \frac{(x-4)^{n+2}}{(n+2)3^{n+1}}, \frac{(x-4)^{n+1}}{(n+1)3^n} \cdot \frac{n3^{n-1}}{(x-4)^n}, \text{ or } \frac{(x-4)^{n+1}}{(n+1)3^n} \cdot \frac{n3^{n-1}}{(x-4)^n}.$$

- **P1** is also earned for ratios mathematically equivalent to the following reciprocal ratios:

$$\frac{(x-4)^{n+1}}{(n+1)3^n} \cdot \frac{(n+2)3^{n+1}}{(x-4)^{n+2}}, \frac{(x-4)^{n+1}}{(n+1)3^n}, \frac{(x-4)^n}{n3^{n-1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}, \text{ or } \frac{(x-4)^n}{n3^{n-1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}}.$$

A response that presents any of these reciprocal ratios earns **P1** and is eligible for **P2**, **P3**, and **P4**, but not **P5**.

- A response that does not present a ratio is not eligible for **P2**.
- A response that does not use the absolute value of a ratio can earn both **P1** and **P2**.
- A response that does not use absolute value in computing the limit is eligible for **P3** if the expression $-1 < \frac{x-4}{3} < 1$ or an equivalent inequality is presented.
- A response that presents an incorrect limit of form $\frac{|x-4|}{b}$, where $b > 0$, is eligible for **P3**.

P3 is then earned for correctly finding an interval with interior $(1, 7)$ or $(4-b, 4+b)$.

Note: $|x-4| < b$ is not sufficient to earn **P3**. $x-4 < b$ can earn **P3** if this is resolved to $-b+4 < x < b+4$.

- **P4** is earned for considering both endpoints of the correct interval or both endpoints of an incorrect interval that has earned **P3**.
- To earn **P5**, a response must correctly analyze the series at $x = 1$ and $x = 7$, and present the correct interval of convergence. Naming of an appropriate test is sufficient for the analysis at each endpoint. In addition to the tests listed in the model solution, the direct comparison test to an appropriate series or the integral test may also be used.

- B** Find the first three nonzero terms and the general term of the Taylor series for f' , the derivative of f , about $x = 4$.

$$f'(x) = \frac{(x-4)}{3} + \frac{(x-4)^2}{9} + \frac{(x-4)^3}{27} + \cdots + \frac{(x-4)^n}{3^n} + \cdots$$

First three terms

Point 6 (P6)

General term

Point 7 (P7)**Scoring Notes for Part B**

- **P6** is earned by presenting the first three nonzero terms in a list or as part of a polynomial or series.
- **P7** is earned by identifying the correct general term (either individually or as part of a polynomial or series).

- C** The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f' , show that $f'(x) = \frac{x-4}{7-x}$.

The Taylor series for f' is a geometric series with first term

$$\frac{x-4}{3} \text{ and common ratio } \frac{x-4}{3}.$$

$$f'(x) = \frac{\frac{x-4}{3}}{1 - \frac{x-4}{3}} = \frac{x-4}{3 - (x-4)} = \frac{x-4}{7-x}$$

Verification

Point 8 (P8)**Scoring Notes for Part C**

- A response of $f'(x) = \frac{\frac{x-4}{3}}{1 - \frac{x-4}{3}}$ is sufficient to earn **P8**.

- D** It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

It follows from the work in part A that the interior of the interval of convergence of the Taylor series for f' is $1 < x < 7$.

Therefore, $x = 8$ would be outside the interval of convergence of f' , and the Taylor series for f' would not converge to

$$f'(x) = \frac{x-4}{7-x} \text{ at } x = 8.$$

Answer with reason **Point 9 (P9)**

Scoring Notes for Part D

- A response of “no, $x = 8$ is outside the interval of convergence” is sufficient to earn **P9**.
- **P9** can be earned with a response consistent with an incorrect interval of convergence imported from part A.
- Alternate solutions:
 - Because the series for $f'(x)$ is geometric, this converges to $f(x)$ for all values of x such that the common ratio $\frac{x-4}{3}$ is between -1 and 1 .

$$-1 < \frac{x-4}{3} < 1 \Rightarrow -3 < x-4 < 3 \Rightarrow 1 < x < 7$$
 - Because $x = 8$ is outside the interval $1 < x < 7$, the series for f' does not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.
 - For $x = 8$, the series is given by $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$, which is a geometric series with $r = \frac{4}{3} > 1$.
 Therefore, the series diverges for $x = 8$.

Part A (AB or BC): Graphing calculator required**Question 1****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f , where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

Model Solution**Scoring**

- (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t) dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t) dt$.

$\int_{60}^{135} f(t) dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.

Interpretation with units **1 point**

$\int_{60}^{135} f(t) dt$
 $\approx f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120)$
 $= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$

Form of Riemann sum **1 point**

Answer **1 point**

Scoring notes:

- To earn the first point the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.

- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30)$, $0.1(30)$, $0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90 - 60) + f(120)(120 - 90) + f(135)(135 - 120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only 8.25 does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9$, or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

Total for part (a) 3 points

(b) Must there exist a value of c , for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

f is differentiable. $\Rightarrow f$ is continuous on $[60, 120]$.

$f(120) - f(60) = 0$ **1 point**

$$\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$$

Answer with justification **1 point**

By the Mean Value Theorem, there must exist a c , for $60 < c < 120$, such that $f'(c) = 0$.

Scoring notes:

- To earn the first point a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must:
 - have earned the first point,
 - state that f is continuous because f is differentiable (or equivalent), and
 - answer “yes” in some way.
- A response may reference either the Mean Value Theorem or Rolle’s Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

Total for part (b) 2 points

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by

$$g(t) = \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right) \text{ for } 0 \leq t \leq 150. \text{ Using this model, find the average rate of flow of}$$

gasoline over the time interval $0 \leq t \leq 150$. Show the setup for your calculations.

$$\frac{1}{150 - 0} \int_0^{150} g(t) dt$$

Average value formula **1 point**

$$= 0.0959967$$

Answer **1 point**

The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).

Scoring notes:

- The exact value of $\frac{1}{150} \int_0^{150} g(t) dt$ is $\frac{12}{125} \sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t) dt = 14.399504$ so the average rate is 0.0959967.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication (e.g., $\int_0^{150} g(t) dt = \frac{14.399504}{150} = 0.0959967$) earns 1 out of 2 points.
- A response of $\int_0^{150} g(t) dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150} \int_0^{150} g(t) dt = 0.149981$ or 0.002618.

Total for part (c) 2 points

- (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

$g'(140) \approx -0.004908$ $g'(140) = -0.005$ (or -0.004)	$g'(140)$ 1 point
The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.	Interpretation 1 point

Scoring notes:

- The exact value of $g'(140)$ is $\frac{1}{500} \cos\left(\frac{49}{36}\right) - \frac{49}{9000} \sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point a response must present some numerical value for $g'(140)$.
- To earn the second point the interpretation must include “the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]” and “at $t = 140$ ” (or equivalent).
- An interpretation of “decreasing at a rate of -0.005 ” or “increasing at a rate of 0.005” does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or 0.00187.

Total for part (d) 2 points**Total for question 1 9 points**