Week 26

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

AP Calculus BC　　Name: ____________________　　Score: ________

1. The value of an infinite series is defined as the limit of its...

   - ☐ terms
   - ☐ partial sums
   - ☐ ratios
   - ☐ derivatives

2. A geometric series $\sum ar^n$ converges exactly when...

   - ☐ $|r| < 1$
   - ☐ $|r| > 1$
   - ☐ $r > 0$
   - ☐ $a < 1$

3. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$...

   - ☐ converges
   - ☐ diverges
   - ☐ could be either
   - ☐ equals 0

4. The Integral Test requires f to be **which** on $[1, \infty)$?

   - ☐ positive
   - ☐ continuous
   - ☐ decreasing
   - ☐ a polynomial

   Tick every correct option.

5. The p-series $\sum \frac{1}{n^p}$ converges exactly when...

   - ☐ $p > 1$
   - ☐ $p < 1$
   - ☐ $p \geq 0$
   - ☐ $p = 1$

6. If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$...

   - ☐ diverges

- ☐ converges
- ☐ is inconclusive
- ☐ equals $\sum b_n$

7. The Alternating Series Test requires the positive terms b_n to be **which**?
   - ☐ decreasing
   - ☐ tending to 0
   - ☐ all equal
   - ☐ a p-series

   Tick every correct option.

8. The partial sums of $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ climb toward what limit?

   ____

9. Approximating $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ with 3 terms, the error bound is the next term b_4. What is it?

   ____

10. For $\sum \frac{x^n}{n}$, the Ratio Test gives $|x| < 1$. What is R?

    ____

AP Calculus BC

1. **partial sums**

   $\sum a_n = \lim_{n\to\infty} S_n$.

2. $|r| < 1$

   Converges iff the ratio is a fraction, $|r| < 1$.

3. **diverges**

   Nonzero term limit $\to$ divergence.

4. **positive; continuous; decreasing**

   Positive, continuous, decreasing — not necessarily a polynomial.

5. $p > 1$

   Converges iff $p > 1$.

6. **converges**

   Smaller than a convergent series $\to$ converges.

7. **decreasing; tending to** 0

   Decreasing and limit 0.

8. **2**

   A geometric series with ratio $\frac{1}{2}$ sums to 2.

9. **0.25**

   $b_4 = \frac{1}{4} = 0.25$.

10. **1**

    $R = 1$.

AP CALCULUS BC • EXERCISE SHEET

10.1 Defining Convergent and Divergent Infinite Series

Name: ________ Class: ________ Date: ________ **Total: 11 marks**

KEY WORDS partial sums 部分和 • converges 收敛 • diverges 发散

Objective

Build the skills to answer exam questions on **convergent and divergent series** — the meaning of an infinite sum.

You must be able to:

- define convergence through the limit of **partial sums** 部分和
- compute the first few partial sums
- decide convergence when the partial sums have a known limit

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Partial sums

The Nth **partial sum** of $\sum a_n$ is $S_N = a_1 + a_2 + \cdots + a_N$. The series **converges** to L if $S_N \to L$; otherwise it **diverges**.

Computing partial sums

For $\sum \left(\frac{1}{2}\right)^n$ starting at $n = 1$: $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$. The partial sums approach 1, so the series converges to 1.

A telescoping series

$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$: $S_N = 1 - \frac{1}{N+1}$, which $\to 1$. Converges to 1.

Divergence by growing sums

For $\sum 1 = 1 + 1 + 1 + \cdots$: $S_N = N \to \infty$, so it diverges. If the partial sums do not settle to a finite value, the series diverges.

2 Practice

Now apply the methods above.

2.1 Write the third partial sum S_3 of $\sum_{n=1}^{\infty} \frac{1}{n}$. [1]

2.2 A series has partial sums $S_N = 3 - \frac{1}{N}$. State its sum. [1]

2.3 A series has $S_N = 2N$. State whether it converges or diverges. [1]

---

3 Exam-style questions

3.1 A series $\sum a_n$ converges if [1]

- **A** $a_n \to 0$
- **B** the partial sums S_N approach a finite limit
- **C** a_n is always positive
- **D** there are infinitely many terms

3.2 For the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$:

(a) Write S_3. [2]

(b) Find a formula for S_N and hence the sum. [3]

3.3 A series has partial sums $S_N = \dfrac{N}{N+1}$. State the sum of the series, with a reason.[2]

Solutions

2.1 $S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

2.2 Sum $= 3$ (as $N \to \infty$, $\frac{1}{N} \to 0$).

2.3 Diverges ($S_N \to \infty$).

3.1 **B** — convergence is defined by the partial sums approaching a finite limit.

3.2 (a) $S_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4} = \frac{3}{4}$. (b) $S_N = 1 - \frac{1}{N+1}$; as $N \to \infty$, sum $= 1$.

3.3 $\dfrac{N}{N+1} \to 1$, so the series converges to 1.

10.2 Working with Geometric Series

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS ratio 比值 • converges 收敛 • geometric series 几何级数 • diverges 发散

Objective

Build the skills to answer exam questions on **geometric series** — the one series you can sum exactly.

You must be able to:

- identify the first term a and common **ratio 比值** r
- apply the convergence test $|r| < 1$
- compute the sum $\dfrac{a}{1-r}$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The geometric sum

A geometric series $\sum_{n=0}^{\infty} ar^n$ converges **exactly when** $|r| < 1$, and then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

■ Reading a and r

For $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$: $a = 3$, $r = \frac{1}{2}$. Since $|r| < 1$, sum $= \dfrac{3}{1-\frac{1}{2}} = 6$.

■ Watch the starting index

If the sum starts at $n = 1$, the first term is ar, not a. $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \dfrac{1/3}{1-1/3} = \dfrac{1/3}{2/3} = \frac{1}{2}$.

■ Divergence

If $|r| \geq 1$ the terms do not shrink, so the series diverges. $\sum 2^n$ has $r = 2$, diverges.

2 Practice

Now apply the methods above.

2.1 State a and r for $\sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n$. [2]

2.2 Find the sum of $\sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n$. [2]

2.3 Does $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$ converge? Give a reason. [1]

3 Exam-style questions

3.1 A geometric series $\sum ar^n$ converges when [1]

- **A** $r > 0$
- **B** $|r| < 1$
- **C** $|r| > 1$
- **D** $a < 1$

3.2 Consider $\sum_{n=0}^{\infty} 6\left(-\frac{1}{2}\right)^n$.

(a) State a and r. [2]

(b) Find the sum. [2]

3.3 Find the sum of $\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$ (note the index starts at 1). [3]

Solutions

2.1 $a = 5$, $r = \frac{1}{3}$.

2.2 $\dfrac{5}{1 - 1/3} = \dfrac{5}{2/3} = \frac{15}{2}$.

2.3 No — $|r| = \frac{5}{4} > 1$, so it diverges.

3.1 **B** — converges iff $|r| < 1$.

3.2 (a) $a = 6$, $r = -\frac{1}{2}$. (b) $\dfrac{6}{1 - (-\frac{1}{2})} = \dfrac{6}{3/2} = 4$.

3.3 First term $(n = 1)$ is $4 \cdot \frac{2}{5} = \frac{8}{5}$, ratio $\frac{2}{5}$; sum $= \dfrac{8/5}{1 - 2/5} = \dfrac{8/5}{3/5} = \frac{8}{3}$.

6. The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$ and converges to $g(x)$

for $|x| < 5$.

Part A

For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Part B

The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Part C

The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \le \frac{1}{5}$.

Part D

The function h is defined by $h(x) = 25g(x) - 2e^x$.

i. Write the first three nonzero terms of the Maclaurin series for e^x.

ii. Write the first three nonzero terms of the Maclaurin series for h.

STOP

END OF EXAM

6. The Taylor series for a function f about $x = 4$ is given by

$$\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2\cdot 3} + \frac{(x-4)^3}{3\cdot 3^2} + \frac{(x-4)^4}{4\cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

STOP

END OF EXAM

AP Calculus BC: **2022**

6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

AP Calculus BC: **2014**

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

STOP

END OF EXAM

AP CALCULUS BC • EXERCISE SHEET

10.3 The nth Term Test for Divergence

Name: ______________ Class: _________ Date: ___________ **Total: 12 marks**

KEY WORDS harmonic series 调和级数 • diverges 发散 • converges 收敛

Objective

Build the skills to answer exam questions on the **nth term test for divergence**.

You must be able to:

- apply: if $\lim_{n\to\infty} a_n \neq 0$, the series **diverges**
- recognise that the test is only a test for **divergence**
- know that $a_n \to 0$ does **not** prove convergence

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The test

If the terms do not shrink to zero, the sum cannot settle:

$$\lim_{n\to\infty} a_n \neq 0 \;\Rightarrow\; \sum a_n \text{ diverges.}$$

A divergent example

$\sum \frac{n}{n+1}$: $\frac{n}{n+1} \to 1 \neq 0$, so the series **diverges** by the nth term test.

It only shows divergence

If $\lim a_n = 0$, the test is **inconclusive** — the series may converge or diverge. The harmonic series $\sum \frac{1}{n}$ has terms $\to 0$ yet **diverges**.

First check to do

Because it is quick, always apply the nth term test **first**. Only if the limit is 0 do you move on to another test.

2 Practice

Now apply the methods above.

2.1 Find $\lim_{n\to\infty} \frac{2n}{3n+1}$. [2]

2.2 State whether $\sum \frac{2n}{3n+1}$ converges or diverges, with a reason. [2]

2.3 For $\sum \frac{1}{n}$, the terms go to 0. Does the nth term test prove convergence? [1]

3 Exam-style questions

3.1 If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ [1]

- **A** converges
- **B** diverges
- **C** may converge or diverge (test inconclusive)
- **D** equals 0

3.2 Consider $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+5}$.

(a) Find $\lim_{n\to\infty} a_n$. [2]

(b) State the conclusion of the nth term test. [2]

3.3 A student claims that because the terms of $\sum \frac{1}{\sqrt{n}}$ tend to 0, the series converges. Explain the error. [2]

Solutions

2.1 $\dfrac{2n}{3n+1} \to \dfrac{2}{3}$.

2.2 Diverges — the limit is $\frac{2}{3} \neq 0$, so by the nth term test it diverges.

2.3 No — the test is inconclusive when the terms go to 0.

3.1 **C** — the test is inconclusive when $a_n \to 0$.

3.2 (a) $\dfrac{n^2}{2n^2+5} \to \dfrac{1}{2}$. (b) Limit $\neq 0$, so the series **diverges**.

3.3 The nth term test only detects **divergence**; $a_n \to 0$ does not prove convergence (in fact $\sum \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$ and diverges).

AP CALCULUS BC • EXERCISE SHEET

10.4 Integral Test for Convergence

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS integral test 积分判别法 • converges 收敛 • diverges 发散 • geometric series 几何级数 • harmonic series 调和级数

Objective

Build the skills to answer exam questions on the **integral test** for convergence.

You must be able to:

- check the conditions (positive, decreasing, continuous)
- compare $\sum a_n$ with $\int_1^{\infty} f(x)\,dx$
- conclude convergence or divergence from the integral

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The integral test

If $a_n = f(n)$ where f is **positive, decreasing, and continuous** for $x \geq 1$, then $\sum a_n$ and $\int_1^{\infty} f(x)\,dx$ **both converge or both diverge**.

■ A worked test

$\sum \frac{1}{n^2}$: with $f(x) = \frac{1}{x^2}$, $\int_1^{\infty} x^{-2}\,dx = 1$ (finite). The integral converges, so the **series converges**.

■ A divergent case

$\sum \frac{1}{n}$: $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b = \infty$. The integral diverges, so the series diverges.

■ Check the conditions first

The function must be positive and decreasing (eventually). The integral's **value** is not the series' sum — only its convergence matches.

2 Practice

Now apply the methods above.

2.1 State the three conditions on f for the integral test. [2]

2.2 Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$. [2]

2.3 From 2.2, does $\sum \frac{1}{n^2}$ converge? [1]

3 Exam-style questions

3.1 The integral test compares a series with [1]

- **A** a geometric series
- **B** an improper integral of the matching function
- **C** the harmonic series
- **D** a partial sum

3.2 Apply the integral test to $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(a) Evaluate $\int_1^\infty \frac{1}{x^3}\,dx$. [3]

(b) State the conclusion. [1]

3.3 Use the integral test to decide whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges. (Use the substitution $u = \ln x$.) [4]

Solutions

2.1 Positive, decreasing, and continuous for $x \geq 1$.

2.2 $\left[-\frac{1}{x}\right]_1^\infty = 0 - (-1) = 1$.

2.3 Yes — the integral converges, so the series converges.

3.1 **B** — with an improper integral of the matching function.

3.2 (a) $\displaystyle\int_1^\infty x^{-3}\,dx = \left[-\frac{1}{2x^2}\right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2}$. (b) Converges.

3.3 $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$; $u = \ln x$, $du = \frac{1}{x}\,dx$; $= \int \frac{1}{u}\,du = \ln|u| = \ln(\ln x)$; as $x \to \infty$ this $\to \infty$, so it **diverges**.

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

(c) Determine the radius of convergence of the Maclaurin series for g.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

10.5 Harmonic Series and p-Series

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS harmonic series 调和级数 • converges 收敛 • diverges 发散

Objective

Build the skills to answer exam questions on the **harmonic series and p-series**.

You must be able to:

- apply the **p-series** rule: $\sum \frac{1}{n^p}$ converges iff $p > 1$
- recognise the **harmonic series 调和级数** ($p = 1$) as divergent
- use p-series as the yardstick for comparison tests

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The p-series rule

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1, \text{ diverges if } p \leq 1.$$

The harmonic series

The special case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series** — it **diverges** even though its terms shrink to 0. A must-know fact.

Applying the rule

- $\sum \frac{1}{n^2}$: $p = 2 > 1$, **converges**.
- $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$: $p = \frac{1}{2} \leq 1$, **diverges**.
- $\sum \frac{1}{n^{3/2}}$: $p = \frac{3}{2} > 1$, **converges**.

The yardstick

p-series are the standard family you compare unknown series to (next subtopic). Know the rule instantly.

2 Practice

Now apply the methods above.

2.1 State whether $\sum \frac{1}{n^4}$ converges. [1]

2.2 State whether $\sum \frac{1}{n^{2/3}}$ converges. [1]

2.3 Does the harmonic series converge? Give the value of p. [2]

3 Exam-style questions

3.1 $\sum \frac{1}{n^p}$ converges if and only if [1]

- **A** $p < 1$
- **B** $p = 1$
- **C** $p > 1$
- **D** $p \geq 0$

3.2 Classify each as convergent or divergent, stating p:

(a) $\sum \frac{1}{n^{1.2}}$ [1]

(b) $\sum \frac{1}{n^{0.9}}$ [1]

(c) $\sum \frac{1}{n}$ [1]

3.3 Explain why $\sum \frac{1}{\sqrt{n}}$ diverges but $\sum \frac{1}{n^2}$ converges, in terms of p. [2]

Solutions

2.1 Converges ($p = 4 > 1$).

2.2 Diverges ($p = \frac{2}{3} \leq 1$).

2.3 No, it diverges; $p = 1$.

3.1 **C** — converges iff $p > 1$.

3.2 (a) convergent, $p = 1.2 > 1$. (b) divergent, $p = 0.9 \leq 1$. (c) divergent, $p = 1$.

3.3 $\sum \frac{1}{\sqrt{n}}$ has $p = \frac{1}{2} \leq 1$ (diverges); $\sum \frac{1}{n^2}$ has $p = 2 > 1$ (converges) — the boundary is $p = 1$.

10.6 Comparison Tests for Convergence

Name: ____________ Class: __________ Date: ___________ **Total: 13 marks**

KEY WORDS limit comparison 极限比较 • direct comparison 直接比较 • converges 收敛 • diverges 发散 • geometric series 几何级数

Objective

Build the skills to answer exam questions on the **comparison tests** for convergence.

You must be able to:

- use the **direct comparison test** with a known series
- use the **limit comparison test** ($\lim a_n/b_n$ finite and positive)
- choose a good comparison series (a p-series or geometric series)

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Direct comparison

If $0 \le a_n \le b_n$ and $\sum b_n$ **converges**, then $\sum a_n$ converges. If $a_n \ge b_n \ge 0$ and $\sum b_n$ **diverges**, then $\sum a_n$ diverges.

A direct comparison example

$\sum \dfrac{1}{n^2+1}$: since $\dfrac{1}{n^2+1} \le \dfrac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges, the series **converges**.

Limit comparison

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ do the **same** thing. Easier when the terms only *behave* like a known series.

A limit comparison example

$\sum \dfrac{2n+1}{n^3+5}$: compare with $b_n = \dfrac{1}{n^2}$. $\dfrac{a_n}{b_n} = \dfrac{(2n+1)n^2}{n^3+5} \to 2$, finite and positive. Since $\sum \frac{1}{n^2}$ converges, so does the series.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{1}{n^3+2}$, name a good comparison series. [1]

2.2 Is $\frac{1}{n^3+2} \leq \frac{1}{n^3}$? What does this tell you? [2]

2.3 For limit comparison of $\sum \frac{n}{n^2+1}$, a good b_n is $\frac{1}{n}$. Find $\lim \frac{a_n}{b_n}$. [2]

3 Exam-style questions

3.1 In the limit comparison test, if $\lim \frac{a_n}{b_n} = L$ with $0 < L < \infty$, then [1]

- **A** $\sum a_n$ always converges
- **B** $\sum a_n$ and $\sum b_n$ behave the same way
- **C** $\sum a_n$ always diverges
- **D** the test is inconclusive

3.2 Use direct comparison to show $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges. [3]

3.3 Use limit comparison with $b_n = \frac{1}{n}$ to decide whether $\sum \dfrac{n+2}{n^2+3}$ converges. [4]

Solutions

2.1 $\sum \dfrac{1}{n^3}$ (a convergent p-series, $p = 3$).

2.2 Yes; since the larger series $\sum \frac{1}{n^3}$ converges, so does $\sum \frac{1}{n^3+2}$.

2.3 $\dfrac{a_n}{b_n} = \dfrac{n/(n^2+1)}{1/n} = \dfrac{n^2}{n^2+1} \to 1$.

3.1 **B** — they converge or diverge together.

3.2 $\dfrac{1}{2^n+1} \leq \dfrac{1}{2^n}$; $\sum \left(\frac{1}{2}\right)^n$ is geometric with $r = \frac{1}{2} < 1$, converges; so by direct comparison the series converges.

3.3 $\dfrac{a_n}{b_n} = \dfrac{(n+2)n}{n^2+3} = \dfrac{n^2+2n}{n^2+3} \to 1$; finite positive, and $\sum \frac{1}{n}$ diverges, so the series **diverges**.

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

(c) Determine the radius of convergence of the Maclaurin series for g.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

10.7 Alternating Series Test for Convergence

Name: ____________ Class: ________ Date: __________ **Total: 11 marks**

KEY WORDS alternating series 交错级数 • converges 收敛 • diverges 发散 • harmonic series 调和级数 • partial sums 部分和

Objective

Build the skills to answer exam questions on the **alternating series test**.

You must be able to:

- recognise an **alternating series** 交错级数 $\sum(-1)^n b_n$
- check the three conditions ($b_n > 0$, decreasing, $\to 0$)
- conclude convergence when they hold

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The alternating series test

An alternating series $\sum(-1)^n b_n$ (with $b_n > 0$) **converges** if:

1. b_n is **decreasing** ($b_{n+1} \le b_n$), and
2. $\lim_{n\to\infty} b_n = 0$.

A worked example

$\sum \dfrac{(-1)^n}{n}$: $b_n = \frac{1}{n}$ is positive, decreasing, and $\to 0$. So the **alternating harmonic series converges** — even though $\sum \frac{1}{n}$ diverges.

Both conditions matter

If the terms b_n do **not** go to 0, the nth term test already forces divergence. If $b_n \to 0$ but is not decreasing, the test does not directly apply.

Recognising the form

The sign alternates via $(-1)^n$ or $(-1)^{n+1}$. Peel off the sign to get b_n, then test b_n.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{(-1)^n}{n^2}$, identify b_n. [1]

2.2 Check the two conditions for $\sum \frac{(-1)^n}{n^2}$. [2]

2.3 State whether $\sum \frac{(-1)^n}{n^2}$ converges. [1]

3 Exam-style questions

3.1 The alternating series test requires that b_n is positive, tends to 0, and is [1]

- **A** increasing
- **B** decreasing
- **C** constant
- **D** geometric

3.2 Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

(a) Identify b_n and check the conditions. [3]

(b) State the conclusion. [1]

3.3 Explain why the alternating harmonic series $\sum \frac{(-1)^n}{n}$ converges even though the harmonic series diverges. [2]

Solutions

2.1 $b_n = \dfrac{1}{n^2}$.

2.2 $b_n = \frac{1}{n^2} > 0$ and decreasing; $\lim b_n = 0$.

2.3 Converges.

3.1 **B** — b_n must be decreasing.

3.2 (a) $b_n = \frac{1}{\sqrt{n}} > 0$, decreasing, and $\to 0$. (b) Converges by the alternating series test.

3.3 The alternating signs let successive terms partly cancel, so the partial sums settle to a limit; without the signs ($\sum \frac{1}{n}$) there is no cancellation and the sum grows without bound.

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\, dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

STOP

END OF EXAM

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

AP CALCULUS BC • EXERCISE SHEET

10.8 Ratio Test for Convergence

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS ratio test 比值判别法 • converges absolutely 绝对收敛 • converges 收敛 • diverges 发散

Objective

Build the skills to answer exam questions on the **ratio test** for convergence.

You must be able to:

- compute $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
- conclude: $L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive
- apply it to series with **factorials** or n**th powers**

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The ratio test

Compute

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

- $L < 1$: **converges absolutely**;
- $L > 1$: **diverges**;
- $L = 1$: **inconclusive**.

■ A factorial example

$\sum \frac{2^n}{n!}$: $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1} \to 0 < 1$. **Converges**.

■ An n-th power example

$\sum \frac{n}{2^n}$: $\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$. **Converges**.

■ When it is inconclusive

For a p-series, $L = 1$, so the ratio test says nothing — use the p-series rule instead. The ratio test shines with factorials and exponentials.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{3^n}{n!}$, write $\frac{a_{n+1}}{a_n}$. [2]

2.2 Find $\lim_{n\to\infty} \frac{3}{n+1}$ and state the conclusion. [2]

2.3 For a p-series, the ratio test gives $L = 1$. What does this mean? [1]

3 Exam-style questions

3.1 If the ratio test gives $L = 1$, the series [1]

- **A** converges
- **B** diverges
- **C** the test is inconclusive
- **D** equals 1

3.2 Apply the ratio test to $\sum_{n=1}^{\infty} \frac{n!}{10^n}$.

(a) Find $\frac{a_{n+1}}{a_n}$ and its limit. [3]

(b) State the conclusion. [1]

3.3 Use the ratio test on $\sum \frac{5^n}{n^2}$. [4]

Solutions

2.1 $\dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}/(n+1)!}{3^n/n!} = \dfrac{3}{n+1}$.

2.2 $\dfrac{3}{n+1} \to 0 < 1$, so the series **converges**.

2.3 The ratio test is inconclusive; use another test (the p-series rule).

3.1 **C** — inconclusive when $L = 1$.

3.2 (a) $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!/10^{n+1}}{n!/10^n} = \dfrac{n+1}{10} \to \infty$. (b) $L > 1$, so it **diverges**.

3.3 $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}/(n+1)^2}{5^n/n^2} = 5 \cdot \dfrac{n^2}{(n+1)^2} \to 5$; $L = 5 > 1$, so it **diverges**.

6. The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM

AP CALCULUS BC • EXERCISE SHEET

10.9 Determining Absolute or Conditional Convergence

Name: ______________ Class: __________ Date: ____________ **Total: 11 marks**

KEY WORDS converges conditionally 条件收敛 • absolute convergence 绝对收敛 • converges 收敛 • diverges 发散 • alternating series 交错级数

Objective

Build the skills to answer exam questions on **absolute and conditional convergence**.

You must be able to:

- test whether $\sum |a_n|$ converges (**absolute convergence 绝对收敛**)
- classify a convergent series as absolutely or **conditionally convergent 条件收敛**
- use the standard example $\sum \dfrac{(-1)^n}{n}$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Absolute convergence

$\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. Absolute convergence is the **stronger** property — it implies the series itself converges.

■ Conditional convergence

If $\sum a_n$ converges but $\sum |a_n|$ diverges, the series **converges conditionally** — it relies on the cancellation of signs.

■ The standard example

$\sum \dfrac{(-1)^n}{n}$ converges (alternating series test), but $\sum \left|\dfrac{(-1)^n}{n}\right| = \sum \dfrac{1}{n}$ diverges. So it is **conditionally** convergent.

■ The routine

1. Test $\sum |a_n|$. If it converges $\rightarrow$ **absolute**. 2. If not, test $\sum a_n$ itself (e.g. alternating series). If it converges $\rightarrow$ **conditional**; if not $\rightarrow$ **divergent**.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{(-1)^n}{n^2}$, does $\sum \left| \frac{(-1)^n}{n^2} \right|$ converge? [1]

2.2 Classify $\sum \frac{(-1)^n}{n^2}$ (absolute / conditional). [1]

2.3 For $\sum \frac{(-1)^n}{n}$, does $\sum \frac{1}{n}$ converge? What does that make the series? [2]

3 Exam-style questions

3.1 A series is **conditionally** convergent if [1]

- **A** both $\sum a_n$ and $\sum |a_n|$ converge
- **B** $\sum a_n$ converges but $\sum |a_n|$ diverges
- **C** both diverge
- **D** $\sum |a_n|$ converges but $\sum a_n$ diverges

3.2 Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

(a) Test $\sum \frac{1}{\sqrt{n}}$ for convergence. [2]

(b) Test the alternating series itself. [1]

(c) Classify the series. [1]

3.3 Explain why absolute convergence is “stronger” than conditional convergence. [2]

Solutions

2.1 Yes — $\sum \frac{1}{n^2}$ is a convergent p-series.

2.2 Absolutely convergent.

2.3 $\sum \frac{1}{n}$ diverges; so $\sum \frac{(-1)^n}{n}$ is **conditionally** convergent.

3.1 **B** — converges, but not absolutely.

3.2 (a) $\sum \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$, diverges. (b) Alternating series test: $b_n = \frac{1}{\sqrt{n}}$ decreasing $\to 0$, so it converges. (c) Conditionally convergent.

3.3 Absolute convergence guarantees the series converges no matter how the terms are arranged, and it implies ordinary convergence; conditional convergence depends on the sign pattern and can be broken by rearrangement.

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

(c) Determine the radius of convergence of the Maclaurin series for g.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\dfrac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \dfrac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

STOP

END OF EXAM

10.10 Alternating Series Error Bound

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

KEY WORDS alternating series 交错级数

Objective

Build the skills to answer exam questions on the **alternating series error bound**.

You must be able to:

- bound the error of a partial sum by the **first omitted term** b_{N+1}
- find how many terms guarantee a required accuracy
- apply $|S - S_N| \le b_{N+1}$ correctly

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The error bound

For a **convergent alternating series**, the error in stopping at S_N is no larger than the **first term you left out**:

$$|S - S_N| \le b_{N+1}.$$

A worked bound

$\sum \dfrac{(-1)^{n+1}}{n}$: using $S_3 = 1 - \frac{1}{2} + \frac{1}{3}$, the error is at most $b_4 = \frac{1}{4} = 0.25$.

How many terms for a target accuracy

To make the error < 0.01, need $b_{N+1} < 0.01$, i.e. $\frac{1}{N+1} < 0.01$, so $N + 1 > 100$, $N \ge 100$. Take 100 terms.

The bound is the next term

The rule is simple but strict: the maximum error equals the **size** of the next term. It only applies to alternating series meeting the alternating-test conditions.

2 Practice

Now apply the methods above.

2.1 State the alternating series error bound. [1]

2.2 For $\sum \frac{(-1)^{n+1}}{n}$ approximated by S_5, bound the error. [2]

2.3 For $\sum \frac{(-1)^{n}}{n^2}$ approximated by S_2, bound the error. [2]

3 Exam-style questions

3.1 For a convergent alternating series, the error $|S - S_N|$ is at most [1]

- **A** b_N
- **B** b_{N+1}
- **C** S_N
- **D** $\sum b_n$

3.2 The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is approximated by its first three terms.

(a) Write the first omitted term. [1]

(b) Give an upper bound for the error. [2]

3.3 How many terms of $\sum \frac{(-1)^{n+1}}{n}$ are needed to guarantee an error less than 0.05? [3]

Solutions

2.1 $|S - S_N| \le b_{N+1}$ (the first omitted term).

2.2 Error $\le b_6 = \frac{1}{6} \approx 0.167$.

2.3 Error $\le b_3 = \frac{1}{9} \approx 0.111$.

3.1 **B** — the error is at most the first omitted term b_{N+1}.

3.2 (a) $b_4 = \dfrac{1}{4^3} = \frac{1}{64}$. (b) Error $\le \frac{1}{64} \approx 0.0156$.

3.3 Need $b_{N+1} = \frac{1}{N+1} < 0.05$; $N + 1 > 20$, so $N \ge 20$; take 20 terms.

6. The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$ and converges to $g(x)$

for $|x| < 5$.

Part A

For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Part B

The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Part C

The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \le \frac{1}{5}$.

Part D

The function h is defined by $h(x) = 25g(x) - 2e^x$.

i. Write the first three nonzero terms of the Maclaurin series for e^x.

ii. Write the first three nonzero terms of the Maclaurin series for h.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

   (a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

   (b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

   (c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

   (d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$

converges absolutely.

(c) Determine the radius of convergence of the Maclaurin series for g.

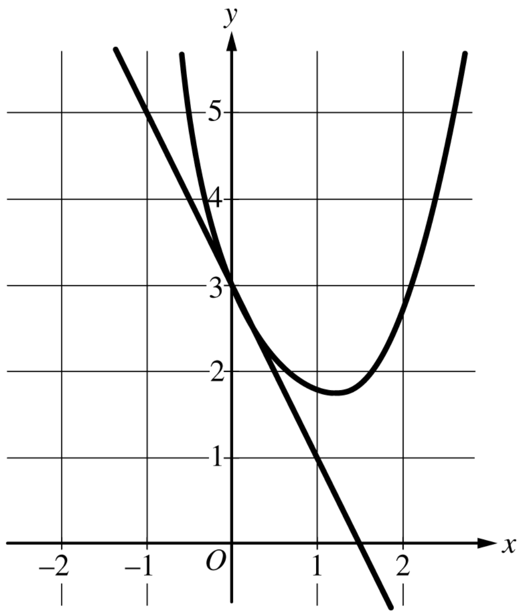
(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating

series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\, dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

STOP

END OF EXAM

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

STOP

END OF EXAM

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

10.11 Finding Taylor Polynomial Approximations of Functions

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS taylor polynomial 泰勒多项式

Objective

Build the skills to answer exam questions on **Taylor polynomials**.

You must be able to:

- build a **Taylor polynomial 泰勒多项式** from derivative values at a center
- use the formula $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
- estimate a function value with the polynomial

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The Taylor polynomial

Centered at $x = a$:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Centered at $a = 0$ it is a **Maclaurin polynomial**.

■ Building from a table

Given $f(0) = 2$, $f'(0) = 3$, $f''(0) = -4$, the 2nd-degree Maclaurin polynomial is

$$P_2(x) = 2 + 3x + \frac{-4}{2!}x^2 = 2 + 3x - 2x^2.$$

■ A worked polynomial for e^x

For $f(x) = e^x$ at $a = 0$: all derivatives are 1, so $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

■ Estimating a value

Use P_n near the center. $e^{0.1} \approx P_3(0.1) = 1 + 0.1 + 0.005 + 0.000167 \approx 1.10517$.

2 Practice

Now apply the methods above.

2.1 Write the general term $\dfrac{f^{(k)}(a)}{k!}(x-a)^k$ for $k=2$. [1]

2.2 Given $f(0)=1,\ f'(0)=2,\ f''(0)=6$, write $P_2(x)$ (Maclaurin). [2]

2.3 For $f(x)=e^x$, write the 2nd-degree Maclaurin polynomial. [2]

3 Exam-style questions

3.1 The coefficient of $(x-a)^2$ in a Taylor polynomial is [1]

- **A** $f''(a)$
- **B** $\dfrac{f''(a)}{2}$
- **C** $2f''(a)$
- **D** $f'(a)$

3.2 A function has $f(1)=3,\ f'(1)=-2,\ f''(1)=4,\ f'''(1)=12$.

(a) Write the 3rd-degree Taylor polynomial about $x=1$. [3]

(b) Use it to estimate $f(1.1)$. [2]

3.3 Write the 4th-degree Maclaurin polynomial for $f(x) = \cos x$ (using $\cos 0 = 1$, and the derivative pattern). [3]

Solutions

2.1 $\dfrac{f''(a)}{2!}(x-a)^2$.

2.2 $P_2(x) = 1 + 2x + \dfrac{6}{2}x^2 = 1 + 2x + 3x^2$.

2.3 $1 + x + \dfrac{x^2}{2}$.

3.1 **B** — the coefficient is $\dfrac{f''(a)}{2!} = \dfrac{f''(a)}{2}$.

3.2 (a) $P_3(x) = 3 - 2(x-1) + \dfrac{4}{2}(x-1)^2 + \dfrac{12}{6}(x-1)^3 = 3 - 2(x-1) + 2(x-1)^2 + 2(x-1)^3$.
(b) at $x = 1.1$: $3 - 0.2 + 2(0.01) + 2(0.001) = 2.822$.

3.3 Derivatives of cos at 0: $1, 0, -1, 0, 1$; $P_4(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$.

5. Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

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6. The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\dfrac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

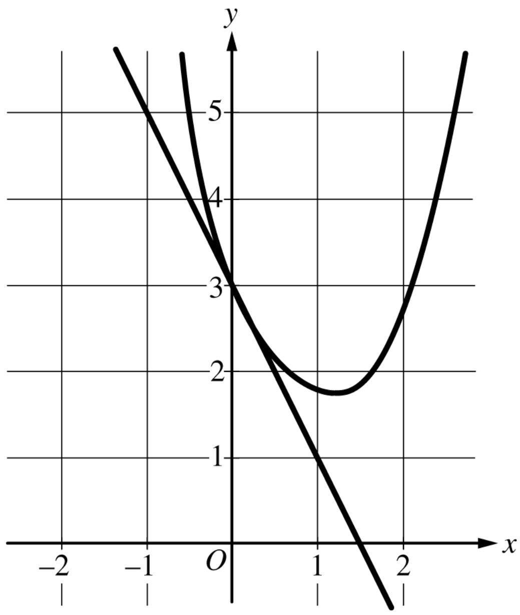
5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

   (a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

   (b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM

10.12 Lagrange Error Bound

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS lagrange error bound 拉格朗日误差界 • taylor polynomial 泰勒多项式

Objective

Build the skills to answer exam questions on the **Lagrange error bound**.

You must be able to:

- apply $|R_n(x)| \le \dfrac{\max|f^{(n+1)}(z)|}{(n+1)!}|x-a|^{n+1}$
- bound the $(n+1)$th derivative on the interval
- compute a numerical error bound

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The Lagrange error bound

The error of the nth Taylor polynomial is bounded by

$$|R_n(x)| \le \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-a|^{n+1},$$

where the max is over z between a and x.

Bounding the derivative

For $f(x) = \sin x$, every derivative is $\pm\sin$ or $\pm\cos$, so $|f^{(n+1)}(z)| \le 1$. This makes the bound easy: replace the max with 1.

A worked bound

Approximating sin(0.2) with the 3rd-degree Maclaurin polynomial ($a = 0$, $n = 3$):

$$|R_3| \le \frac{1}{4!}|0.2|^4 = \frac{0.0016}{24} \approx 6.7 \times 10^{-5}.$$

■ **The routine**

1. Find $\max|f^{(n+1)}|$ on the interval. 2. Multiply by $\dfrac{|x-a|^{n+1}}{(n+1)!}$.

2 Practice

Now apply the methods above.

2.1 State the Lagrange error bound. [1]

2.2 For $f(x) = \cos x$, state a bound for $|f^{(n+1)}(z)|$. [1]

2.3 Compute $\dfrac{1}{3!}(0.1)^3$. [2]

3 Exam-style questions

3.1 In the Lagrange error bound, the factor $(n+1)!$ appears in the [1]

- **A** numerator
- **B** denominator
- **C** exponent
- **D** derivative

3.2 The function $f(x) = e^x$ is approximated near 0 by its 2nd-degree Maclaurin polynomial, for $0 \le x \le 0.5$. On this interval $|f'''(z)| \le e^{0.5} < 2$.

(a) Write the Lagrange bound for $|R_2(0.5)|$. [2]

(b) Evaluate the bound using $|f'''| \le 2$. [2]

3.3 Bound the error when $\sin(0.3)$ is approximated by the 1st-degree Maclaurin polynomial $P_1(x) = x$. [3]

Solutions

2.1 $|R_n(x)| \leq \dfrac{\max|f^{(n+1)}(z)|}{(n+1)!}|x-a|^{n+1}$.

2.2 $|f^{(n+1)}(z)| \leq 1$ (derivatives of cos are bounded by 1).

2.3 $\dfrac{1}{6}(0.001) = \dfrac{0.001}{6} \approx 1.67 \times 10^{-4}$.

3.1 **B** — $(n+1)!$ is in the denominator.

3.2 (a) $|R_2(0.5)| \leq \dfrac{\max|f'''|}{3!}(0.5)^3$. (b) $\leq \dfrac{2}{6}(0.125) = \dfrac{0.25}{6} \approx 0.0417$.

3.3 $n = 1$, $|f''(z)| = |-\sin z| \leq 1$; $|R_1(0.3)| \leq \dfrac{1}{2!}(0.3)^2 = \dfrac{0.09}{2} = 0.045$.

6. The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots \text{ and converges to } g(x)$$

for $|x| < 5$.

Part A

For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Part B

The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Part C

The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \le \frac{1}{5}$.

Part D

The function h is defined by $h(x) = 25g(x) - 2e^x$.

i. Write the first three nonzero terms of the Maclaurin series for e^x.

ii. Write the first three nonzero terms of the Maclaurin series for h.

STOP

END OF EXAM

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

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6. The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

10.13 Radius and Interval of Convergence of Power Series

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS interval of convergence 收敛区间 • radius of convergence 收敛半径 • converges 收敛 • ratio test 比值判别法 • diverges 发散

Objective

Build the skills to answer exam questions on the **radius and interval of convergence** of a power series.

You must be able to:

- use the **ratio test** to find the **radius of convergence** 收敛半径 R
- test the **endpoints** separately
- state the full **interval of convergence** 收敛区间

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Radius by the ratio test

A power series $\sum c_n(x-a)^n$ converges for $|x-a| < R$. Find R by applying the ratio test and requiring $L < 1$.

A worked radius

$\sum \dfrac{(x-2)^n}{3^n}$: $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x-2|}{3} \to \dfrac{|x-2|}{3}$. Set < 1: $|x-2| < 3$, so $R = 3$, center 2: the open interval is $(-1, 5)$.

Test the endpoints

The ratio test is inconclusive at $x = -1$ and $x = 5$. At $x = 5$: $\sum 1$ diverges; at $x = -1$: $\sum(-1)^n$ diverges. So the interval is the open $(-1, 5)$.

Stating the interval

Combine the open interval with each endpoint's separate result, using [ ] (included) or ( ) (excluded). Always test **both** ends.

2 Practice

Now apply the methods above.

2.1 For $\sum \frac{x^n}{2^n}$, apply the ratio test to find R. [3]

2.2 State the open interval of convergence for a series centered at 0 with $R = 2$. [1]

2.3 Why must the endpoints be tested separately? [1]

3 Exam-style questions

3.1 The radius of convergence is found using the [1]

- **A** integral test
- **B** ratio test
- **C** nth term test
- **D** comparison test

3.2 Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

(a) Find the radius of convergence. [3]

(b) Test the endpoints $x = 0$ and $x = 2$. [2]

(c) State the interval of convergence. [1]

3.3 Find the radius of convergence of $\sum \frac{x^n}{n!}$. [3]

Solutions

2.1 $\left|\dfrac{x^{n+1}/2^{n+1}}{x^n/2^n}\right| = \dfrac{|x|}{2}$; set < 1: $|x| < 2$; $R = 2$.

2.2 $(-2, 2)$.

2.3 The ratio test gives $L = 1$ at the endpoints, so it is inconclusive there.

3.1 **B** — the ratio test gives the radius.

3.2 (a) $\left|\dfrac{(x-1)^{n+1}/(n+1)}{(x-1)^n/n}\right| = |x-1| \cdot \dfrac{n}{n+1} \to |x-1|; < 1 \Rightarrow R = 1$. (b) $x = 2$: $\sum \frac{1}{n}$ diverges; $x = 0$: $\sum \frac{(-1)^n}{n}$ converges. (c) $[0, 2)$.

3.3 $\left|\dfrac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \dfrac{|x|}{n+1} \to 0 < 1$ for all x; $R = \infty$ (converges everywhere).

6. The Taylor series for a function f about $x=4$ is given by

$\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x=4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x=4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x=4$ is the same as the radius of convergence of the Taylor series for f' about $x=4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x=8$? Give a reason for your answer.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$

converges absolutely.

(c) Determine the radius of convergence of the Maclaurin series for g.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating

series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1 + \frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2) - f(2)\right|$.

STOP

END OF EXAM

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1} (-1)^{n+1} \frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

   (a) Find the value of R.

   (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

   (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

STOP

END OF EXAM

10.14 Finding Taylor or Maclaurin Series for a Function

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS geometric series 几何级数

Objective

Build the skills to answer exam questions on **finding Taylor and Maclaurin series**.

You must be able to:

- recall the key **Maclaurin series** for e^x, $\sin x$, $\cos x$, $\dfrac{1}{1-x}$
- build new series by **substitution** into a known one
- write a series in summation form

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The must-know Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Building by substitution

For e^{x^2}, substitute x^2 into the e^x series:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Another substitution

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

■ **Writing out terms**

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$. Being able to write the first few terms **and** the general summation form is expected.

2 Practice

Now apply the methods above.

2.1 Write the Maclaurin series for e^x (summation form). [1]

2.2 Write the first three nonzero terms of $\cos x$. [2]

2.3 Use substitution to write the Maclaurin series for e^{-x}. [2]

3 Exam-style questions

3.1 The Maclaurin series for $\frac{1}{1-x}$ is [1]

- **A** $\sum(-1)^n x^n$
- **B** $\sum x^n$
- **C** $\sum \frac{x^n}{n!}$
- **D** $\sum nx^n$

3.2 Find the Maclaurin series for $\dfrac{1}{1+x^2}$ (substitute into the geometric series). [3]

3.3 Find the first three nonzero terms of the Maclaurin series for $x \sin x$. [3]

Solutions

2.1 $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.2 $1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$.

2.3 $\displaystyle\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$.

3.1 **B** — $\dfrac{1}{1-x} = \sum x^n$.

3.2 $\displaystyle\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

3.3 $\sin x = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \cdots$; multiply by x: $x^2 - \dfrac{x^4}{6} + \dfrac{x^6}{120} - \cdots$.

6. The Maclaurin series for a function g is given by
$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$ and converges to $g(x)$ for $|x| < 5$.

Part A

For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Part B

The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Part C

The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \leq \frac{1}{5}$.

Part D

The function h is defined by $h(x) = 25g(x) - 2e^x$.

i. Write the first three nonzero terms of the Maclaurin series for e^x.

ii. Write the first three nonzero terms of the Maclaurin series for h.

STOP

END OF EXAM

6. The Taylor series for a function f about $x=4$ is given by $\sum_{n=1}^{\infty}\frac{(x-4)^{n+1}}{(n+1)3^n}=\frac{(x-4)^2}{2\cdot 3}+\frac{(x-4)^3}{3\cdot 3^2}+\frac{(x-4)^4}{4\cdot 3^3}+\cdots+\frac{(x-4)^{n+1}}{(n+1)3^n}+\cdots$ and converges to $f(x)$ on its interval of convergence.

A. Using the ratio test, find the interval of convergence of the Taylor series for f about $x=4$. Justify your answer.

B. Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x=4$.

C. The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x)=\frac{x-4}{7-x}$.

D. It is known that the radius of convergence of the Taylor series for f about $x=4$ is the same as the radius of convergence of the Taylor series for f' about $x=4$. Does the Taylor series for f' described in part B converge to $f'(x)=\frac{x-4}{7-x}$ at $x=8$? Give a reason for your answer.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

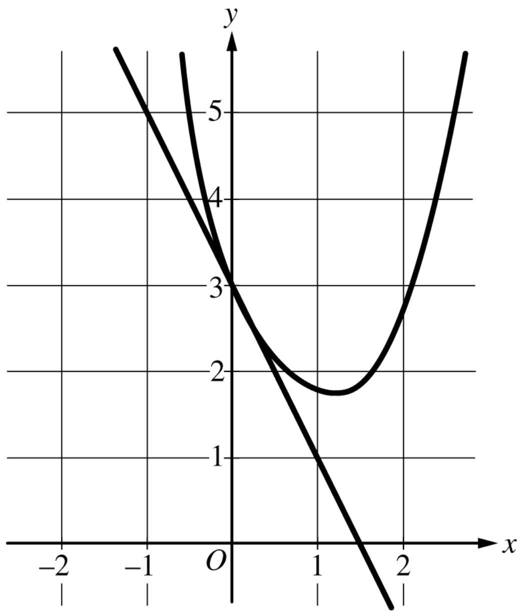
(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM



n	$f^{(n)}(0)$
2	3
3	$-\dfrac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

6. The Maclaurin series for $\ln(1 + x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1 + x)$. Let f be the function defined by $f(x) = x\ln\left(1 + \frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2) - f(2)\right|$.

STOP

END OF EXAM

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\, dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

STOP

END OF EXAM

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

   (c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

   (d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

   (a) Find the value of R.

   (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

   (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

STOP

END OF EXAM

10.15 Representing Functions as Power Series

Name: ____________ Class: __________ Date: __________ **Total: 14 marks**

KEY WORDS converges 收敛 • power series 幂级数 • radius of convergence 收敛半径 • geometric series 几何级数

Objective

Build the skills to answer exam questions on **representing functions as power series** — building new series by calculus.

You must be able to:

- **differentiate** or **integrate** a known power series term by term
- build a new series from the geometric series
- use a series to approximate an integral

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Term-by-term calculus

Within its radius of convergence, a power series can be **differentiated** and **integrated** term by term to get the series of the derivative or antiderivative.

■ Building a new series by integrating

Start from $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$. Integrating gives

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

■ Building by substitution then integrating

$\dfrac{1}{1+x^2} = \sum(-1)^n x^{2n}$. Integrating term by term gives the series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

■ **Approximating an integral**

A function like e^{-x^2} has no elementary antiderivative, but its **series** integrates term by term, letting you approximate $\int_0^1 e^{-x^2}\,dx$.

2 Practice

Now apply the methods above.

2.1 Differentiate the series $\sum_{n=0}^{\infty} x^n$ term by term. [2]

2.2 Write the Maclaurin series for $\frac{1}{1-x}$ and integrate its first three terms. [2]

2.3 State the first three nonzero terms of the series for $\arctan x$. [1]

3 Exam-style questions

3.1 Within its radius of convergence, a power series may be [1]

- **A** differentiated but not integrated
- **B** integrated but not differentiated
- **C** both differentiated and integrated term by term
- **D** neither

3.2 Start from $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Write the series for e^{-x^2}. [2]

(b) Write the first three terms of $\int_0^1 e^{-x^2}\, dx$ obtained by integrating that series term by term. [3]

3.3 Use the geometric series to write a power series for $\frac{1}{1-2x}$, and state its radius of convergence. [3]

Solutions

2.1 $\dfrac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1}$.

2.2 $\sum x^n = 1 + x + x^2 + \cdots$; integrating: $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$.

2.3 $x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$.

3.1 **C** — both, term by term, within the radius.

3.2 (a) $e^{-x^2} = \sum_{n=0}^{\infty} \dfrac{(-x^2)^n}{n!} = \sum \dfrac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \dfrac{x^4}{2} - \cdots$. (b) $\int_0^1 \big(1 - x^2 + \frac{x^4}{2} - \cdots\big)dx = \big[x - \frac{x^3}{3} + \frac{x^5}{10} - \cdots\big]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \cdots$.

3.3 $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$; converges for $|2x| < 1$, i.e. $R = \frac{1}{2}$.

6. The Maclaurin series for a function g is given by

$$\sum_{n=0}^{\infty} 2\left(\frac{-x}{5}\right)^n = 2 - \frac{2}{5}x + \frac{2}{25}x^2 - \frac{2}{125}x^3 + \frac{2}{625}x^4 - \cdots + 2\left(\frac{-x}{5}\right)^n + \cdots$$ and converges to $g(x)$

for $|x| < 5$.

Part A

For $x > 0$, the Maclaurin series for g is an alternating geometric series. Find $g(3)$. Show the work that leads to your answer.

Part B

The function f is defined by $f(x) = g'(x)$. Write the first four nonzero terms of the Maclaurin series for f.

Part C

The second-degree Taylor polynomial for f about $x = 0$ is used to approximate $f\left(\frac{5}{2}\right)$ as $-\frac{3}{10}$, where f is the function defined in part B. Justify that $\left|f\left(\frac{5}{2}\right) - \left(-\frac{3}{10}\right)\right| \leq \frac{1}{5}$.

Part D

The function h is defined by $h(x) = 25g(x) - 2e^x$.

i. Write the first three nonzero terms of the Maclaurin series for e^x.

ii. Write the first three nonzero terms of the Maclaurin series for h.

STOP

END OF EXAM

6. The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.

(b) Show that $\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{10}$. Justify your answer.

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

STOP

END OF EXAM

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

STOP

END OF EXAM

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM