Week 22

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

9. Parametric Equations, Polar Coordinates, and Vector-Valued Functions
Starter Quiz · 课前小测

AP Calculus BC | Name: ______________ | Score: ______

1. For a parametric curve, $\dfrac{dy}{dx} =$
   - ☐ $\dfrac{dy/dt}{dx/dt}$
   - ☐ $\dfrac{dx/dt}{dy/dt}$
   - ☐ $\dfrac{dy}{dt} \cdot \dfrac{dx}{dt}$
   - ☐ $\dfrac{d}{dt}\left[\frac{y}{x}\right]$

2. The parametric second derivative $\dfrac{d^2y}{dx^2}$ equals...
   - ☐ $\dfrac{d^2y/dt^2}{d^2x/dt^2}$
   - ☐ $\dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$
   - ☐ $\dfrac{dy/dt}{dx/dt}$
   - ☐ $\dfrac{d^2y}{dt^2}$

3. The arc length of a parametric curve is $\int_a^b (\ ?\ )\, dt$ where the integrand is...
   - ☐ $\sqrt{1 + (dy/dx)^2}$
   - ☐ $\sqrt{(dx/dt)^2 + (dy/dt)^2}$
   - ☐ $(dx/dt) + (dy/dt)$
   - ☐ $\sqrt{(dx/dt)^2 - (dy/dt)^2}$

4. To differentiate $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, you...
   - ☐ multiply the components
   - ☐ differentiate each component separately
   - ☐ add the components then differentiate
   - ☐ take the magnitude first

5. To integrate $\langle f(t), g(t)\rangle$, you...

   - ☐ integrate each component separately
   - ☐ multiply the components
   - ☐ take the magnitude
   - ☐ differentiate instead

6. For $\mathbf{v}(t) = \langle 2t, 3\rangle$, find the speed at $t = 2$.

   ____

7. For $x = t^2$, $y = t^3$, $\dfrac{dy}{dx} = \dfrac{3t}{2}$. Find the slope at $t = 2$.

   ____

8. The parametric second derivative equals $\dfrac{d^2y/dt^2}{d^2x/dt^2}$.

   ☐ True ☐ False

9. The parametric arc-length integrand equals the point's ______.

   ____

10. Integrating a vector function introduces a constant vector $\mathbf{C}$.

    ☐ True ☐ False

AP Calculus BC

1. $\dfrac{dy/dt}{dx/dt}$

   Top over bottom: $\frac{dy/dt}{dx/dt}$.

2. $\dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$

   Differentiate $\frac{dy}{dx}$ in t, then divide by $\frac{dx}{dt}$.

3. $\sqrt{(dx/dt)^2 + (dy/dt)^2}$

   Both derivatives squared, added, under a root.

4. **differentiate each component separately**

   $\mathbf{r}' = \langle x', y' \rangle$.

5. **integrate each component separately**

   Integrate componentwise, plus a constant vector.

6. **5**

   $\sqrt{4^2 + 3^2} = 5$.

7. **3**

   $\frac{3(2)}{2} = 3$.

8. **False**

   That naive ratio is wrong.

9. **speed**

   Integral of speed = distance.

10. **True**

    One constant per component, bundled into $\mathbf{C}$.

AP CALCULUS BC • EXERCISE SHEET

9.1 Defining and Differentiating Parametric Equations

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS parametric equations 参数方程

Objective

Build the skills to answer exam questions on **parametric equations and their derivatives**.

You must be able to:

- describe a curve by **parametric equations** 参数方程 $x = x(t)$, $y = y(t)$
- find the slope $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$
- evaluate the slope at a given t

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Parametric curves

A parametric curve gives x and y each as a function of a parameter t: as t varies, the point $(x(t), y(t))$ traces the curve. Example: $x = t^2$, $y = t^3$.

The slope formula

The chain rule gives the slope of the curve:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad (dx/dt \neq 0).$$

A worked slope

For $x = t^2$, $y = t^3$: $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$, so

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}.$$

At $t = 2$ the slope is 3.

■ **Horizontal and vertical tangents**

A **horizontal** tangent needs $\frac{dy}{dt} = 0$ (and $\frac{dx}{dt} \neq 0$); a **vertical** tangent needs $\frac{dx}{dt} = 0$.

2 Practice

Now apply the methods above.

2.1 For $x = t^2$, $y = t^3$, state $\frac{dx}{dt}$ and $\frac{dy}{dt}$. [2]

2.2 For $x = \cos t$, $y = \sin t$, find $\frac{dy}{dx}$. [2]

2.3 For $x = t^2$, $y = 4t$, find the slope at $t = 1$. [2]

3 Exam-style questions

3.1 For a parametric curve, $\frac{dy}{dx} =$ [1]

- **A** $\frac{dx/dt}{dy/dt}$
- **B** $\frac{dy/dt}{dx/dt}$
- **C** $\frac{dy}{dt} \cdot \frac{dx}{dt}$
- **D** $\frac{d^2y}{dt^2}$

3.2 A curve is given by $x = t^2 - 1$, $y = t^3 - 3t$.

(a) Find $\frac{dy}{dx}$ in terms of t. [2]

(b) Find the value(s) of t where the tangent is horizontal. [2]

3.3 For $x = e^t$, $y = e^{2t}$, find the slope of the tangent line at $t = 0$. [3]

Solutions

2.1 $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$.

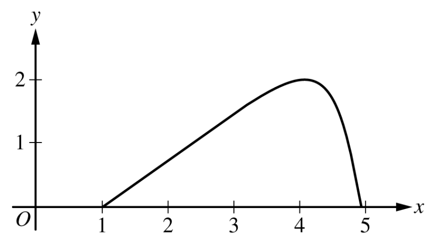
2.2 $\dfrac{dy}{dx} = \dfrac{\cos t}{-\sin t} = -\cot t$.

2.3 $\dfrac{dy}{dx} = \dfrac{4}{2t} = \dfrac{2}{t}$; at $t = 1$ slope $= 2$.

3.1 B — $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

3.2 (a) $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t}$. (b) horizontal where $3t^2 - 3 = 0 \Rightarrow t = \pm 1$.

3.3 $\dfrac{dx}{dt} = e^t$, $\dfrac{dy}{dt} = 2e^{2t}$; $\dfrac{dy}{dx} = \dfrac{2e^{2t}}{e^t} = 2e^t$; at $t = 0$, slope $= 2$.



2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\dfrac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

   (a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

   (b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

   (c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

   (d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

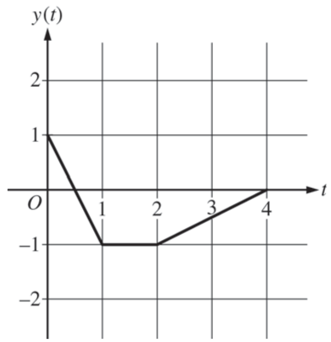
CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t),\ y(t))$, where $\dfrac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.2 Second Derivatives of Parametric Equations

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS concavity 凹凸性

Objective

Build the skills to answer exam questions on the **second derivative of a parametric curve** — testing concavity.

You must be able to:

- compute $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$
- avoid the wrong "divide the second derivatives" shortcut
- use the sign of $\dfrac{d^2y}{dx^2}$ to state **concavity** 凹凸性

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The correct formula

The second derivative is **not** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t, then divide by $\dfrac{dx}{dt}$ again:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

A worked example

For $x = t^2$, $y = t^3$: $\dfrac{dy}{dx} = \dfrac{3t}{2}$. Then $\dfrac{d}{dt}\left(\dfrac{3t}{2}\right) = \dfrac{3}{2}$, and $\dfrac{dx}{dt} = 2t$, so

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}.$$

Concavity

At $t = 1$, $\dfrac{d^2y}{dx^2} = \frac{3}{4} > 0$: the curve is **concave up** there. A negative value means concave down.

The two-step routine

1. Find $\dfrac{dy}{dx}$ (a function of t). 2. Differentiate it w.r.t. t and divide by $\dfrac{dx}{dt}$.

2 Practice

Now apply the methods above.

2.1 State the formula for $\dfrac{d^2y}{dx^2}$ for a parametric curve. [1]

2.2 For $x = t^2$, $y = t^3$, given $\dfrac{dy}{dx} = \dfrac{3t}{2}$, find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$. [1]

2.3 Complete: find $\dfrac{d^2y}{dx^2}$ for that curve. [2]

3 Exam-style questions

3.1 The second derivative of a parametric curve is [1]

- **A** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$
- **B** $\dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$
- **C** $\dfrac{dy/dt}{dx/dt}$
- **D** $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$

3.2 A curve has $x = t$, $y = t^3 - 3t$.

(a) Find $\dfrac{dy}{dx}$. [1]

(b) Find $\dfrac{d^2y}{dx^2}$. [2]

(c) State the concavity at $t = 1$. [1]

3.3 For $x = t^2$, $y = t^2 + t$, find $\dfrac{d^2y}{dx^2}$ at $t = 1$. [4]

Solutions

2.1 $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$.

2.2 $\dfrac{d}{dt}\left(\dfrac{3t}{2}\right) = \dfrac{3}{2}$.

2.3 $\dfrac{d^2y}{dx^2} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$.

3.1 **B** — differentiate dy/dx w.r.t. t, then divide by dx/dt.

3.2 (a) $\dfrac{dy}{dx} = \dfrac{3t^2-3}{1} = 3t^2 - 3$. (b) $\dfrac{d}{dt}(3t^2-3) = 6t$; divide by $dx/dt = 1$: $\dfrac{d^2y}{dx^2} = 6t$. (c) at $t = 1$, $6 > 0$: concave up.

3.3 $\dfrac{dy}{dx} = \dfrac{2t+1}{2t}$; $\dfrac{d}{dt}\left(\dfrac{2t+1}{2t}\right) = \dfrac{d}{dt}(1 + \frac{1}{2t}) = -\dfrac{1}{2t^2}$; divide by $2t$: $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4t^3}$; at $t = 1$: $-\frac{1}{4}$.

AP CALCULUS BC • EXERCISE SHEET

9.3 Finding Arc Lengths of Curves Given by Parametric Equations

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS parametric equations 参数方程

Objective

Build the skills to answer exam questions on the **arc length of a parametric curve**.

You must be able to:

- apply $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- set up the integral from parametric equations
- recognise it as summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The parametric arc-length formula

For $x = x(t)$, $y = y(t)$ with t from a to b:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Setting it up

For $x = t^2$, $y = t^3$ on $[0,1]$: $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$, so

$$L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.$$

A curve that simplifies

For a circle $x = \cos t$, $y = \sin t$: $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, so the integrand is $\sqrt{\sin^2 t + \cos^2 t} = 1$. Then $L = \int_0^{2\pi} 1\, dt = 2\pi$ (the circumference).

Same idea as before

This is the $\sqrt{dx^2 + dy^2}$ hypotenuse summed along the curve, now with both x and y depending on t.

2 Practice

Now apply the methods above.

2.1 State the parametric arc-length formula. [1]

2.2 For $x = t^2$, $y = t^3$, write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$. [2]

2.3 For $x = 3t$, $y = 4t$ on $[0, 2]$, set up and evaluate the arc length. [3]

3 Exam-style questions

3.1 The parametric arc length uses the integrand [1]

- **A** $\sqrt{1 + (dy/dx)^2}$
- **B** $\sqrt{(dx/dt)^2 + (dy/dt)^2}$
- **C** $\dfrac{dy/dt}{dx/dt}$
- **D** $(dx/dt)(dy/dt)$

3.2 A curve is $x = \cos t$, $y = \sin t$ for $0 \le t \le \pi$.

(a) Show the integrand simplifies to 1. [2]

(b) Evaluate the arc length. [1]

3.3 Set up (do not evaluate) the arc-length integral for $x = t - \sin t$, $y = 1 - \cos t$ on $0 \le t \le 2\pi$. [3]

Solutions

2.1 $L = \displaystyle\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$.

2.2 $(2t)^2 + (3t^2)^2 = 4t^2 + 9t^4$.

2.3 $\sqrt{9+16} = 5$; $L = \displaystyle\int_0^2 5\,dt = 10$.

3.1 **B** — the parametric integrand is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.

3.2 (a) $(-\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1$. (b) $\int_0^\pi 1\,dt = \pi$.

3.3 $\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$; $L = \displaystyle\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\,dt$.

2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

   (a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

   (b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

   (c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

   (d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

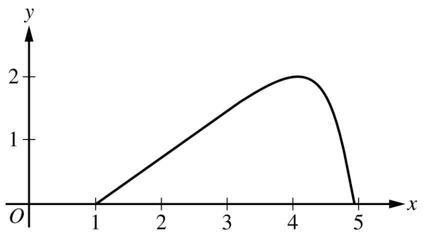
CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\dfrac{dx}{dt} = \sqrt{1+t^2}$ and $\dfrac{dy}{dt} = \ln\left(2+t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

   (a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

   (b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

   (c) Find the y-coordinate of the particle's position at time $t = 6$.

   (d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

AP Calculus BC: **2021**

2. For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

   (a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

   (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

   (c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

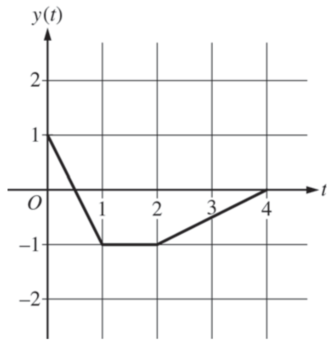
CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.4 Defining and Differentiating Vector-Valued Functions

Name: ______ Class: ______ Date: ______ **Total: 15 marks**

KEY WORDS vector-valued function 向量值函数

Objective

Build the skills to answer exam questions on **vector-valued functions** — differentiating a position vector.

You must be able to:

- read a **vector-valued function 向量值函数** $\vec{r}(t) = \langle x(t), y(t)\rangle$
- differentiate **component-wise** to get velocity and acceleration
- find the **speed** $|\vec{v}| = \sqrt{x'^2 + y'^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Position, velocity, acceleration

A vector function gives a position for each t: $\vec{r}(t) = \langle x(t), y(t)\rangle$. Differentiate each component:

$$\vec{v}(t) = \langle x'(t), y'(t)\rangle, \qquad \vec{a}(t) = \langle x''(t), y''(t)\rangle.$$

A worked derivative

For $\vec{r}(t) = \langle t^2, \sin t\rangle$: $\vec{v}(t) = \langle 2t, \cos t\rangle$ and $\vec{a}(t) = \langle 2, -\sin t\rangle$.

Speed is a magnitude

The **speed** is the length of the velocity vector:

$$|\vec{v}| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}.$$

For $\vec{v} = \langle 3, 4\rangle$, speed $= \sqrt{9+16} = 5$.

Evaluating at a point

At a specific t, plug in to get the numeric velocity, acceleration, and speed. At $t = 0$ for $\vec{r} = \langle t^2, \sin t\rangle$: $\vec{v}(0) = \langle 0, 1\rangle$, speed $= 1$.

2 Practice

Now apply the methods above.

2.1 For $\vec{r}(t) = \langle t^3, t^2\rangle$, find $\vec{v}(t)$. [2]

2.2 For $\vec{v}(t) = \langle 2t, 3\rangle$, find the speed at $t = 2$. [2]

2.3 For $\vec{r}(t) = \langle \cos t, \sin t\rangle$, find $\vec{a}(t)$. [2]

3 Exam-style questions

3.1 The speed of a particle with velocity $\vec{v} = \langle x', y'\rangle$ is [1]

- **A** $x' + y'$
- **B** $\sqrt{x'^2 + y'^2}$
- **C** $\langle x', y'\rangle$
- **D** $x'y'$

3.2 A particle has position $\vec{r}(t) = \langle t^2 - 1, 2t\rangle$.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$. [2]

(b) Find the speed at $t = 1$. [2]

3.3 For $\vec{r}(t) = \langle e^t, e^{-t} \rangle$, find the velocity and speed at $t = 0$. [4]

Solutions

2.1 $\vec{v}(t) = \langle 3t^2, 2t \rangle$.

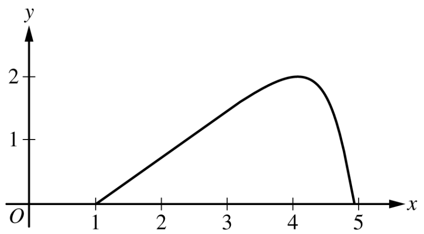
2.2 $\vec{v}(2) = \langle 4, 3 \rangle$; speed $= \sqrt{16+9} = 5$.

2.3 $\vec{v} = \langle -\sin t, \cos t \rangle$; $\vec{a} = \langle -\cos t, -\sin t \rangle$.

3.1 B — speed is the magnitude $\sqrt{x'^2 + y'^2}$.

3.2 (a) $\vec{v} = \langle 2t, 2 \rangle$, $\vec{a} = \langle 2, 0 \rangle$. (b) $\vec{v}(1) = \langle 2, 2 \rangle$; speed $= \sqrt{4+4} = 2\sqrt{2}$.

3.3 $\vec{v} = \langle e^t, -e^{-t} \rangle$; at $t = 0$, $\vec{v} = \langle 1, -1 \rangle$; speed $= \sqrt{1+1} = \sqrt{2}$.



2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\dfrac{dx}{dt} = \sqrt{1+t^2}$ and $\dfrac{dy}{dt} = \ln\left(2+t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

   (a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

   (b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

   (c) Find the y-coordinate of the particle's position at time $t = 6$.

   (d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

AP Calculus BC: **2021**

2. For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

   (a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

   (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

   (c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.5 Integrating Vector-Valued Functions

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS vector-valued function 向量值函数

Objective

Build the skills to answer exam questions on **integrating vector-valued functions**.

You must be able to:

- integrate a vector function **component by component**
- use an **initial condition** to find each constant
- recover velocity from acceleration, or position from velocity

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Integrate each component

$\int \vec{r}(t)\,dt$ integrates each component separately, giving a **vector** of constants $\vec{C}$:

$$\int \langle f(t), g(t)\rangle\,dt = \left\langle \int f\,dt, \int g\,dt \right\rangle + \vec{C}.$$

■ From acceleration to velocity

Given $\vec{a}(t) = \langle 2, -6t\rangle$ and $\vec{v}(0) = \langle 1, 0\rangle$: integrate, $\vec{v}(t) = \langle 2t + C_1, -3t^2 + C_2\rangle$. Apply $\vec{v}(0) = \langle 1, 0\rangle$: $C_1 = 1$, $C_2 = 0$, so $\vec{v}(t) = \langle 2t + 1, -3t^2\rangle$.

■ From velocity to position

Integrate $\vec{v}$ and use $\vec{r}(0)$ to fix the constants. Each component is a separate initial-value problem.

■ Definite integral of a vector

$\int_a^b \vec{v}(t)\,dt$ gives the **displacement** vector over $[a, b]$ — integrate each component between the limits.

2 Practice

Now apply the methods above.

2.1 Find $\int \langle 2t, 3\rangle\,dt$ (include the constant vector). [2]

2.2 Given $\vec{a}(t) = \langle 0, -10\rangle$ and $\vec{v}(0) = \langle 5, 0\rangle$, find $\vec{v}(t)$. [3]

2.3 Evaluate $\int_0^2 \langle 1, 2t\rangle\,dt$. [2]

3 Exam-style questions

3.1 To integrate a vector-valued function you [1]

- **A** integrate each component separately
- **B** take the magnitude first
- **C** differentiate it
- **D** add the components then integrate

3.2 A particle has acceleration $\vec{a}(t) = \langle 6t, 2\rangle$, with $\vec{v}(0) = \langle 0, 1\rangle$.

(a) Find $\vec{v}(t)$. [3]

(b) Find $\vec{v}(1)$. [1]

3.3 A particle has velocity $\vec{v}(t) = \langle 2t, \cos t\rangle$ and starts at $\vec{r}(0) = \langle 1, 0\rangle$. Find the position $\vec{r}(t)$. [4]

Solutions

2.1 $\langle t^2,\, 3t\rangle + \vec{C}$.

2.2 $\vec{v}(t) = \langle C_1, -10t + C_2\rangle$; $\vec{v}(0) = \langle 5, 0\rangle \Rightarrow C_1 = 5, C_2 = 0$; $\vec{v}(t) = \langle 5, -10t\rangle$.

2.3 $\left\langle [t]_0^2, [t^2]_0^2 \right\rangle = \langle 2, 4\rangle$.

3.1 **A** — integrate each component separately.

3.2 (a) $\vec{v}(t) = \langle 3t^2 + C_1, 2t + C_2\rangle$; $\vec{v}(0) = \langle 0, 1\rangle \Rightarrow C_1 = 0, C_2 = 1$; $\vec{v}(t) = \langle 3t^2, 2t + 1\rangle$.
(b) $\vec{v}(1) = \langle 3, 3\rangle$.

3.3 $\vec{r}(t) = \langle t^2 + C_1, \sin t + C_2\rangle$; $\vec{r}(0) = \langle 1, 0\rangle \Rightarrow C_1 = 1, C_2 = 0$; $\vec{r}(t) = \langle t^2 + 1, \sin t\rangle$.

2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1+t^2}$ and $\frac{dy}{dt} = \ln\left(2+t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

(b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

(c) Find the y-coordinate of the particle's position at time $t = 6$.

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

AP Calculus BC: **2015**

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on **planar motion** with parametric and vector functions.

You must be able to:

- find velocity, acceleration, and **speed** from a position vector
- find the **position at a later time** (start + $\int \vec{v}\,dt$)
- find the **total distance travelled** $\int_a^b \sqrt{x'^2 + y'^2}\,dt$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The motion toolkit

For a particle at $\vec{r}(t) = \langle x(t), y(t)\rangle$: velocity $\vec{v} = \langle x', y'\rangle$, acceleration $\vec{a} = \langle x'', y''\rangle$, and speed $|\vec{v}| = \sqrt{x'^2 + y'^2}$.

Position at a later time

If you know $\vec{r}(a)$ and $\vec{v}(t)$, the position at b is

$$\vec{r}(b) = \vec{r}(a) + \int_a^b \vec{v}(t)\,dt,$$

component by component.

Total distance travelled

The **distance travelled** (not displacement) over $[a, b]$ is

$$\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$$

— the parametric arc length of the path.

A worked speed

For $\vec{v}(t) = \langle 3t, 4t\rangle$: speed $= \sqrt{9t^2 + 16t^2} = 5t$; at $t = 2$, speed $= 10$.

2 Practice

Now apply the methods above.

2.1 For $\vec{v}(t) = \langle 6, 8\rangle$, find the speed. [1]

2.2 A particle has $\vec{v}(t) = \langle 2t, 2\rangle$. Set up the total distance travelled over $0 \le t \le 3$. [2]

2.3 A particle starts at $\vec{r}(0) = \langle 1, 2\rangle$ with $\vec{v}(t) = \langle 4, 6t\rangle$. Find $\vec{r}(t)$. [3]

3 Exam-style questions

3.1 The total distance a particle travels over $[a, b]$ is [1]

- **A** $\int_a^b \vec{v}\,dt$
- **B** $|\vec{r}(b) - \vec{r}(a)|$
- **C** $\int_a^b \sqrt{x'^2 + y'^2}\,dt$
- **D** $\vec{r}(b) - \vec{r}(a)$

3.2 A particle moves with $\vec{v}(t) = \langle 3t^2, 2t \rangle$, starting at $\vec{r}(0) = \langle 0, 1 \rangle$.

(a) Find the position $\vec{r}(t)$. [3]

(b) Find the speed at $t = 1$. [2]

3.3 A particle has $\vec{v}(t) = \langle \cos t, \sin t \rangle$. Set up and evaluate the total distance travelled over $0 \leq t \leq \pi$. [3]

Solutions

2.1 $\sqrt{36 + 64} = 10$.

2.2 $\displaystyle\int_0^3 \sqrt{(2t)^2 + 2^2}\, dt = \int_0^3 \sqrt{4t^2 + 4}\, dt$.

2.3 $\vec{r}(t) = \langle 4t + 1, 3t^2 + 2 \rangle$.

3.1 **C** — total distance is the arc-length integral of speed.

3.2 (a) $\vec{r}(t) = \langle t^3, t^2 + 1 \rangle$. (b) $\vec{v}(1) = \langle 3, 2 \rangle$; speed $= \sqrt{9 + 4} = \sqrt{13}$.

3.3 speed $= \sqrt{\cos^2 t + \sin^2 t} = 1$; distance $= \displaystyle\int_0^\pi 1\, dt = \pi$.

2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

   (a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

   (b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

   (c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

   (d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

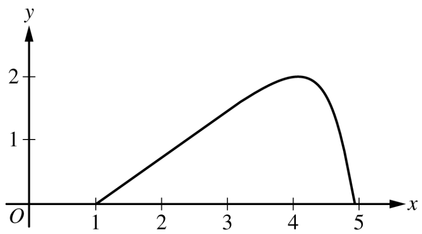
CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\frac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1+t^2}$ and $\frac{dy}{dt} = \ln\left(2+t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

   (a) Find the slope of the line tangent to the path of the particle at time $t = 4$.

   (b) Find the speed of the particle at time $t = 4$, and find the acceleration vector of the particle at time $t = 4$.

   (c) Find the y-coordinate of the particle's position at time $t = 6$.

   (d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

AP Calculus BC: **2021**

2. For time $t \geq 0$, a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

   (a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

   (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

   (c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \le h \le 30$ and is modeled by $f(h)$ for $h \ge 30$. The continuous function f is not explicitly given.

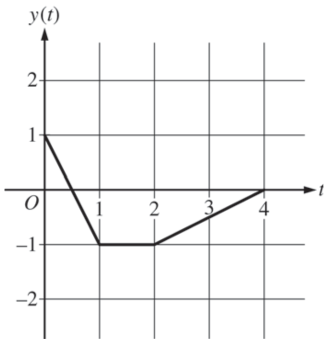
(a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

(b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

(c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\,dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

(d) The boat is moving on the surface of the sea. At time $t \ge 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \le t \le 1$.

END OF PART A OF SECTION II



2. At time t, the position of a particle moving in the xy-plane is given by the parametric functions $(x(t), y(t))$, where $\frac{dx}{dt} = t^2 + \sin(3t^2)$. The graph of y, consisting of three line segments, is shown in the figure above. At $t = 0$, the particle is at position $(5, 1)$.

(a) Find the position of the particle at $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at $t = 3$.

(c) Find the speed of the particle at $t = 3$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

END OF PART A OF SECTION II

2. At time $t \geq 0$, a particle moving along a curve in the xy-plane has position $(x(t), y(t))$ with velocity vector $v(t) = \left(\cos\left(t^2\right), e^{0.5t}\right)$. At $t = 1$, the particle is at the point $(3, 5)$.

(a) Find the x-coordinate of the position of the particle at time $t = 2$.

(b) For $0 < t < 1$, there is a point on the curve at which the line tangent to the curve has a slope of 2. At what time is the object at that point?

(c) Find the time at which the speed of the particle is 3.

(d) Find the total distance traveled by the particle from time $t = 0$ to time $t = 1$.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.7 Defining Polar Coordinates and Differentiating in Polar Form

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS polar 极坐标

Objective

Build the skills to answer exam questions on **polar coordinates** and differentiating in polar form.

You must be able to:

- convert between **polar** 极坐标 (r, θ) and Cartesian (x, y)
- treat $r = f(\theta)$ as a parametric curve $x = r\cos\theta$, $y = r\sin\theta$
- find $\dfrac{dy}{dx}$ for a polar curve

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Polar to Cartesian

A point (r, θ) has $x = r\cos\theta$, $y = r\sin\theta$. So $(2, \frac{\pi}{3})$ is $x = 2\cos\frac{\pi}{3} = 1$, $y = 2\sin\frac{\pi}{3} = \sqrt{3}$.

■ A polar curve is parametric in θ

For $r = f(\theta)$, use θ as the parameter:

$$x = f(\theta)\cos\theta, \qquad y = f(\theta)\sin\theta.$$

■ Slope of a polar curve

The slope is the parametric slope with θ as parameter:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Compute $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ from the product rule.

■ A worked conversion

The circle $r = 4$ is $x = 4\cos\theta$, $y = 4\sin\theta$ — the circle of radius 4. The line $\theta = \frac{\pi}{4}$ is $y = x$ (for $r > 0$).

2 Practice

Now apply the methods above.

2.1 Convert the polar point $(3, \frac{\pi}{2})$ to Cartesian coordinates. [2]

2.2 Write x and y for the polar curve $r = 2\cos\theta$. [2]

2.3 Convert the Cartesian point $(0, 5)$ to polar coordinates. [2]

3 Exam-style questions

3.1 For a polar curve $r = f(\theta)$, $x =$ [1]

- **A** $r\sin\theta$
- **B** $r\cos\theta$
- **C** $\dfrac{r}{\cos\theta}$
- **D** $r + \cos\theta$

3.2 For the polar curve $r = 1 + \cos\theta$:

(a) Write x and y as functions of θ. [2]

(b) Find the point (in Cartesian coordinates) at $\theta = 0$. [2]

3.3 For $r = 2\sin\theta$, find $\dfrac{dx}{d\theta}$. [3]

Solutions

2.1 $x = 3\cos\frac{\pi}{2} = 0$, $y = 3\sin\frac{\pi}{2} = 3$: $(0, 3)$.

2.2 $x = 2\cos\theta\cos\theta = 2\cos^2\theta$, $y = 2\cos\theta\sin\theta$.

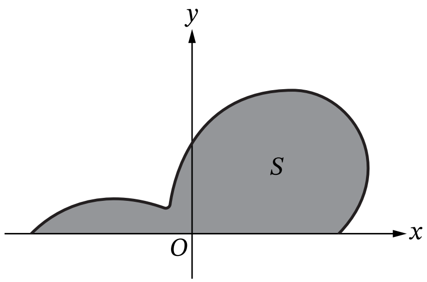
2.3 $r = 5$, $\theta = \frac{\pi}{2}$: $(5, \frac{\pi}{2})$.

3.1 **B** — $x = r\cos\theta$.

3.2 (a) $x = (1+\cos\theta)\cos\theta$, $y = (1+\cos\theta)\sin\theta$. (b) at $\theta = 0$: $r = 2$, $x = 2$, $y = 0$: $(2, 0)$.

3.3 $x = 2\sin\theta\cos\theta = \sin(2\theta)$; $\dfrac{dx}{d\theta} = 2\cos(2\theta)$. (Or via product rule on $2\sin\theta\cos\theta$.)

2. Let S be the shaded region bounded by the graph of the polar curve $r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ for $0 \le \theta \le \pi$, as shown in the figure.



It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Part A

Find the area of region S. Show the setup for your calculations.

Part B

There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Part C

i. Find the value of θ in the interval $0 < \theta < \frac{\pi}{2}$ at which r has a critical point.

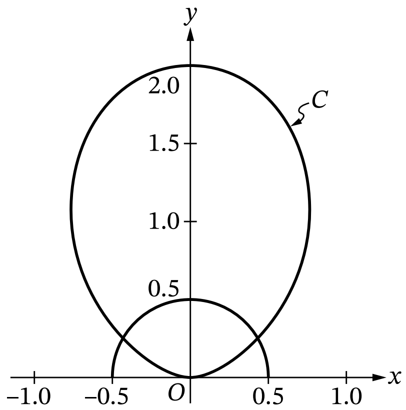
ii. Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r.

Part D

Find the average distance from the origin to a point on the polar curve $r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ for $\frac{\pi}{2} \le \theta \le \pi$. Show the setup for your calculations.

END OF PART A

2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

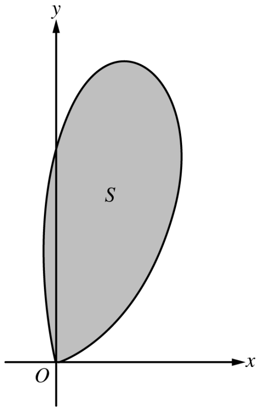
A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.

B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

C. It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

D. A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

END OF PART A



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

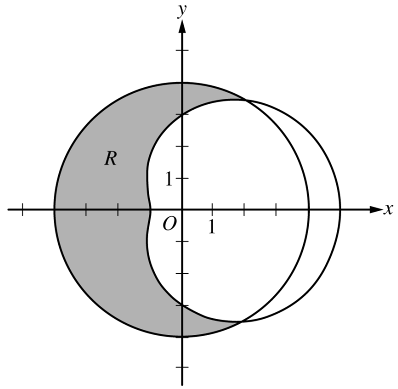
(a) Find the area of S.

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim\limits_{k\to\infty} A(k)$.

END OF PART A OF SECTION II

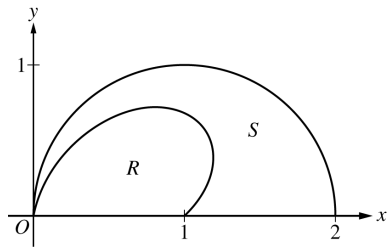


5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

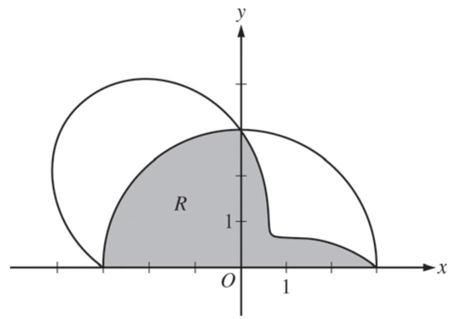
(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

END OF PART A OF SECTION II



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.8 Finding the Area of a Polar Region

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **area of a polar region**.

You must be able to:

- apply $A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta$
- choose the correct θ-limits that trace the region once
- set up the area for a full curve or a sector

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The polar area formula

The area swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta.$$

The region is a fan of thin triangular sectors of area $\frac{1}{2}r^2 \, d\theta$.

■ A worked area

The circle $r = 2$: $A = \frac{1}{2}\int_{0}^{2\pi} 2^2 \, d\theta = \frac{1}{2}(4)(2\pi) = 4\pi$ — matching πr^2.

■ Choosing the limits

For $r = 2\cos\theta$ (a circle traced once as θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$), integrate over exactly that range so the curve is traced **once**.

■ One petal of a rose

For $r = \sin(2\theta)$, one petal is traced as θ goes from 0 to $\frac{\pi}{2}$. Its area is $\frac{1}{2}\int_{0}^{\pi/2} \sin^2(2\theta) \, d\theta$.

2 Practice

Now apply the methods above.

2.1 State the polar area formula. [1]

2.2 Set up the area enclosed by $r = 3$ (a full circle). [2]

2.3 Evaluate $\frac{1}{2}\int_{0}^{2\pi} 9 \, d\theta$. [2]

3 Exam-style questions

3.1 The area of a polar region is [1]

- **A** $\int_{\alpha}^{\beta} f(\theta) \, d\theta$
- **B** $\frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2 \, d\theta$
- **C** $\pi \int_{\alpha}^{\beta} f(\theta) \, d\theta$
- **D** $\frac{1}{2}\int_{\alpha}^{\beta} f(\theta) \, d\theta$

3.2 Consider $r = 1 + \cos\theta$ (a cardioid), traced once for $0 \le \theta \le 2\pi$.

(a) Set up the area integral. [2]

(b) Expand $(1 + \cos\theta)^2$ ready to integrate. [2]

3.3 One petal of the rose $r = 2\sin(3\theta)$ is traced as θ goes from 0 to $\frac{\pi}{3}$. Set up the area integral for one petal. [3]

Solutions

2.1 $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \big(f(\theta)\big)^2\, d\theta.$

2.2 $A = \dfrac{1}{2}\displaystyle\int_{0}^{2\pi} 3^2\, d\theta.$

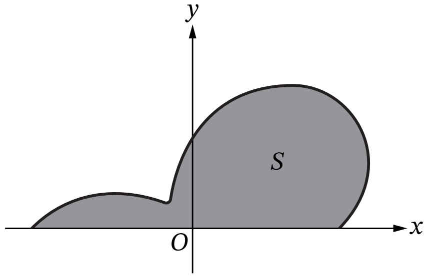
2.3 $\dfrac{1}{2}(9)(2\pi) = 9\pi.$

3.1 **B** — half the integral of r^2.

3.2 (a) $A = \dfrac{1}{2}\displaystyle\int_{0}^{2\pi} (1 + \cos\theta)^2\, d\theta.$ (b) $(1 + \cos\theta)^2 = 1 + 2\cos\theta + \cos^2\theta.$

3.3 $A = \dfrac{1}{2}\displaystyle\int_{0}^{\pi/3} \big(2\sin(3\theta)\big)^2\, d\theta = \dfrac{1}{2}\displaystyle\int_{0}^{\pi/3} 4\sin^2(3\theta)\, d\theta.$

2. Let S be the shaded region bounded by the graph of the polar curve $r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ for $0 \le \theta \le \pi$, as shown in the figure.



It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Part A

Find the area of region S. Show the setup for your calculations.

Part B

There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Part C

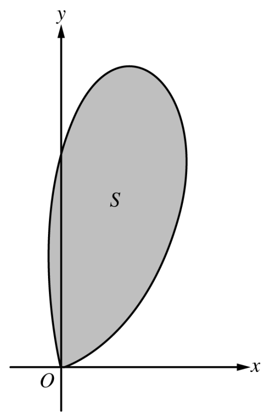
i. Find the value of θ in the interval $0 < \theta < \frac{\pi}{2}$ at which r has a critical point.

ii. Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r.

Part D

Find the average distance from the origin to a point on the polar curve $r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ for $\frac{\pi}{2} \le \theta \le \pi$. Show the setup for your calculations.

END OF PART A



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

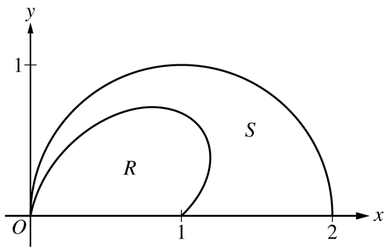
(a) Find the area of S.

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim\limits_{k\to\infty} A(k)$.

END OF PART A OF SECTION II



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

END OF PART A OF SECTION II

AP CALCULUS BC • EXERCISE SHEET

9.9 Finding the Area of the Region Bounded by Two Polar Curves

Name: ____________ Class: __________ Date: ____________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **area between two polar curves**.

You must be able to:

- find the **intersection angles** of two polar curves ($r_1 = r_2$)
- integrate $\frac{1}{2}\left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right)$
- decide which curve is outer over each interval

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The formula

The area between an outer curve r_1 and an inner curve r_2 (over the same θ-range) is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right) d\theta.$$

Subtract the inner sector from the outer sector.

Finding intersection angles

Set the curves equal. For $r = 3$ and $r = 2 + 2\cos\theta$: $3 = 2 + 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$.

Which is outer?

Test a θ between the intersections. Between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, at $\theta = 0$: $r = 3$ vs $r = 4$, so the cardioid $2 + 2\cos\theta$ is outer there.

Setting up

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((2 + 2\cos\theta)^2 - 3^2\right) d\theta.$$

Use symmetry to write it as $2 \times \frac{1}{2}\int_0^{\pi/3}(\dots)\,d\theta$ if convenient.

2 Practice

Now apply the methods above.

2.1 State the area-between-two-polar-curves formula. [1]

2.2 Find the intersection angles of $r = 2$ and $r = 4\cos\theta$. [3]

2.3 Between $\theta = 0$ and $\theta = \frac{\pi}{6}$, which is larger: $r = 2$ or $r = 4\cos\theta$? [1]

3 Exam-style questions

3.1 The area between an outer curve r_1 and inner r_2 is [1]

- **A** $\frac{1}{2}\int (r_1 - r_2)^2\, d\theta$
- **B** $\frac{1}{2}\int (r_1^2 - r_2^2)\, d\theta$
- **C** $\int (r_1^2 - r_2^2)\, d\theta$
- **D** $\frac{1}{2}\int (r_1^2 + r_2^2)\, d\theta$

3.2 Two curves are $r = 1$ (inner) and $r = 2\cos\theta$ (outer) where they overlap.

(a) Find the intersection angles. [2]

(b) Set up the area inside $r = 2\cos\theta$ and outside $r = 1$. [3]

3.3 Explain why you must square the radii **before** subtracting, i.e. why $r_1^2 - r_2^2 \neq (r_1 - r_2)^2$ in the area formula. [2]

Solutions

2.1 $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right)\, d\theta.$

2.2 $2 = 4\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}.$

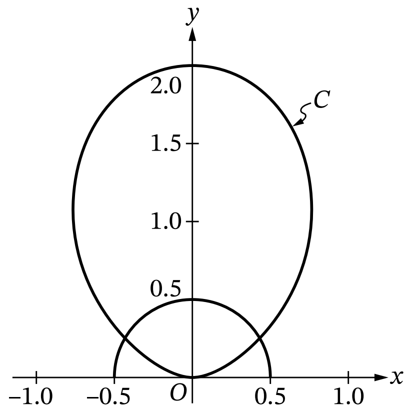
2.3 $r = 4\cos\theta$ is larger (at $\theta = 0$: $4 > 2$).

3.1 **B** — half the integral of the difference of the squared radii.

3.2 (a) $1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$. (b) $A = \dfrac{1}{2}\displaystyle\int_{-\pi/3}^{\pi/3} \left((2\cos\theta)^2 - 1^2\right) d\theta.$

3.3 Each sector's area is $\frac{1}{2}r^2\, d\theta$, so the region between is $\frac{1}{2}(r_1^2 - r_2^2)\, d\theta$ — a difference of the two **squared** radii; $(r_1 - r_2)^2$ would wrongly square the difference of the radii instead.

2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

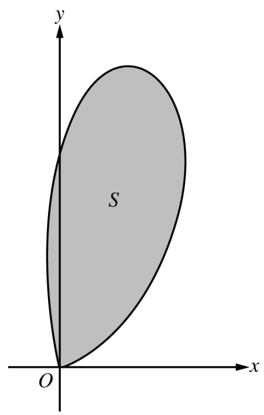
A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.

B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

C. It can be shown that $\dfrac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

D. A particle travels along curve C so that $\dfrac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

END OF PART A



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

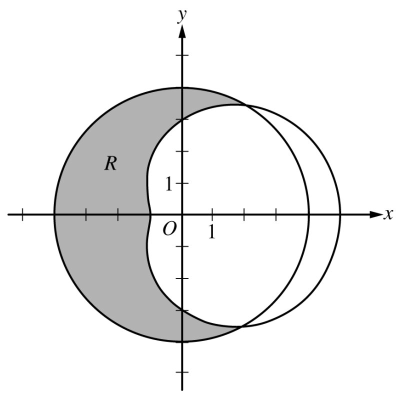
   (a) Find the area of S.

   (b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

   (c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

   (d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim\limits_{k\to\infty} A(k)$.

END OF PART A OF SECTION II

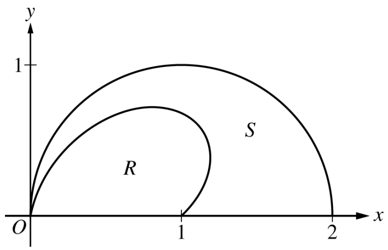


5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

   (a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

   (b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

   (c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

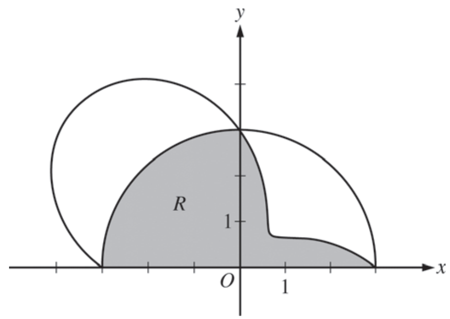
(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ? Give a reason for your answer.

END OF PART A OF SECTION II



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

END OF PART A OF SECTION II