Week 21

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

8. Applications of Integration

Starter Quiz · 课前小测

AP Calculus BC Name: ____________ Score: ______

1. The average value of f on $[a, b]$ is...
   - ☐ $\int_a^b f\,dx$
   - ☐ $\dfrac{1}{b-a}\int_a^b f\,dx$
   - ☐ $(b-a)\int_a^b f\,dx$
   - ☐ $f(b) - f(a)$
2. To recover position from velocity you...
   - ☐ differentiate velocity
   - ☐ integrate velocity
   - ☐ square velocity
   - ☐ take the reciprocal
3. The applied accumulation relationship is: final value =...
   - ☐ $\int_a^b R\,dt$
   - ☐ initial value $+ \int_a^b R\,dt$
   - ☐ initial value $\times \int_a^b R\,dt$
   - ☐ $R(b) - R(a)$
4. The area between an upper curve f and lower curve g on $[a, b]$ is...
   - ☐ $\int_a^b (g - f)\,dx$
   - ☐ $\int_a^b (f - g)\,dx$
   - ☐ $\int_a^b f\,dx$
   - ☐ $\int_a^b (f \cdot g)\,dx$
5. With horizontal slices, the area is $\int_c^d (\ ?\ )\,dy$ where...
   - ☐ top minus bottom
   - ☐ rightmost minus leftmost
   - ☐ leftmost minus rightmost
   - ☐ the product of the curves
6. Where do $y = x^3$ and $y = x$ cross?

☐ $x = -1$

☐ $x = 0$

☐ $x = 1$

☐ $x = 2$

Tick every correct option.

7. Find the average value of $f(x) = x^2$ on $[0, 3]$. ($\int_0^3 x^2\,dx = 9$.)

____

8. For $v(t) = t - 2$ on $[0, 3]$, find the displacement $\int_0^3 (t-2)\,dt$.

____

9. A tank has 50 L; water flows in at $R(t) = 4t$ for $0 \le t \le 3$ ($\int_0^3 4t\,dt = 18$). Find the total at $t = 3$.

____

10. Where do $y = x + 2$ and $y = x^2$ intersect?

☐ $x = -1$

☐ $x = 2$

☐ $x = 0$

☐ $x = 1$

Tick every correct option.

AP Calculus BC

1. $\dfrac{1}{b-a}\int_a^b f\,dx$

   Total divided by width.

2. **integrate velocity**

   Position = initial + $\int v\,dt$.

3. **initial value** $+\int_a^b R\,dt$

   Final = initial + accumulated change.

4. $\int_a^b (f-g)\,dx$

   Upper minus lower, integrated.

5. **rightmost minus leftmost**

   Horizontal strips: right curve minus left curve.

6. $x=-1$; $x=0$; $x=1$

   $x^3 = x \Rightarrow x(x-1)(x+1)=0$.

7. **3**

   $\frac{1}{3}\cdot 9 = 3$.

8. **-1.5**

   $\frac{9}{2} - 6 = -\frac{3}{2}$.

9. **68**

   $50 + 18 = 68$ L.

10. $x=-1$; $x=2$

    $x^2 - x - 2 = 0 \Rightarrow x = -1, 2$.

8.1 Finding the Average Value of a Function on an Interval

Name: ______ Class: ______ Date: ______ **Total: 16 marks**

KEY WORDS average value 平均值 • total distance 总路程

Objective

Build the skills to answer exam questions on the **average value of a function** — the integral divided by the width.

You must be able to:

- apply $f_{\text{avg}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$
- distinguish the **average value** 平均值 of a function from the average **rate of change**
- interpret the result in context

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The average-value formula

The average value of f over $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It is the constant height of a rectangle with the same area as the region under f.

■ A worked value

Average of $f(x) = x^2$ on $[0, 3]$: $\dfrac{1}{3-0}\displaystyle\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}\cdot 9 = 3.$

■ Average value vs average rate of change

- **Average value** of f uses an **integral**: $\dfrac{1}{b-a}\int_a^b f$.
- **Average rate of change** of f uses a **difference**: $\dfrac{f(b)-f(a)}{b-a}$.

Read the question carefully to pick the right one.

■ **In context**

If $v(t)$ is speed, its average value $\frac{1}{b-a}\int_a^b v\,dt$ is the **average speed** over the interval — total distance divided by time.

2 Practice

Now apply the methods above.

2.1 Write the formula for the average value of f on $[2, 6]$. [1]

2.2 Find the average value of $f(x) = 4x$ on $[0, 5]$. [3]

2.3 Find the average value of $f(x) = 6x^2$ on $[0, 2]$. [3]

3 Exam-style questions

3.1 The average value of f on $[a, b]$ is [1]

- **A** $f(b) - f(a)$
- **B** $\dfrac{f(b) - f(a)}{b - a}$
- **C** $\dfrac{1}{b-a}\displaystyle\int_a^b f\,dx$
- **D** $\displaystyle\int_a^b f\,dx$

3.2 A tank's inflow rate is $r(t) = 12 - t$ litres/min for $0 \leq t \leq 8$.

(a) Find the average inflow rate over this interval. [3]

(b) State the units of your answer. [1]

3.3 The temperature over a day is modelled by $T(t) = 20 + 6t - t^2$ for $0 \leq t \leq 6$ hours. Find the average temperature over the interval. [4]

Solutions

2.1 $\dfrac{1}{6-2}\displaystyle\int_2^6 f(x)\,dx.$

2.2 $\dfrac{1}{5}\displaystyle\int_0^5 4x\,dx = \dfrac{1}{5}\big[2x^2\big]_0^5 = \dfrac{1}{5}(50) = 10.$

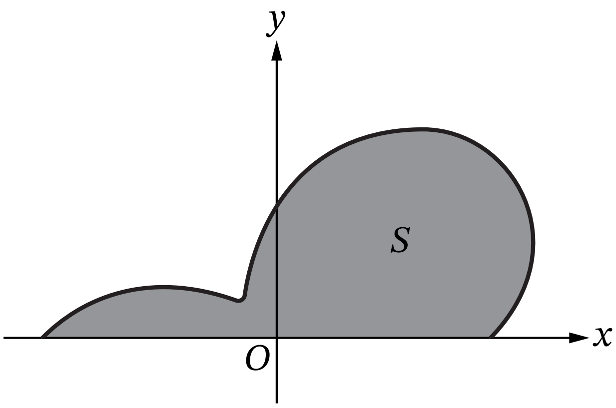
2.3 $\dfrac{1}{2}\displaystyle\int_0^2 6x^2\,dx = \dfrac{1}{2}\big[2x^3\big]_0^2 = \dfrac{1}{2}(16) = 8.$

3.1 C — the average value is the integral divided by the width.

3.2 (a) $\dfrac{1}{8}\displaystyle\int_0^8 (12-t)\,dt = \dfrac{1}{8}\big[12t - \tfrac{t^2}{2}\big]_0^8 = \dfrac{1}{8}(96-32) = 8.$ (b) litres per minute.

3.3 $\dfrac{1}{6}\displaystyle\int_0^6 (20+6t-t^2)\,dt = \dfrac{1}{6}\big[20t + 3t^2 - \tfrac{t^3}{3}\big]_0^6 = \dfrac{1}{6}(120+108-72) = \dfrac{1}{6}(156) = 26°.$

2. Let S be the shaded region bounded by the graph of the polar curve $r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ for $0 \le \theta \le \pi$, as shown in the figure.



It can be shown that $r'(\theta) = 4\cos(2\theta) - 2\sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Part A

Find the area of region S. Show the setup for your calculations.

Part B

There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Part C

i. Find the value of θ in the interval $0 < \theta < \frac{\pi}{2}$ at which r has a critical point.

ii. Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r.

Part D

Find the average distance from the origin to a point on the polar curve $r(\theta) = 3 + 2\sin(2\theta) + \cos(2\theta)$ for $\frac{\pi}{2} \le \theta \le \pi$. Show the setup for your calculations.

END OF PART A

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

   (c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

   (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

   Show the setup for your calculations.

   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

   (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

   (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

   (d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

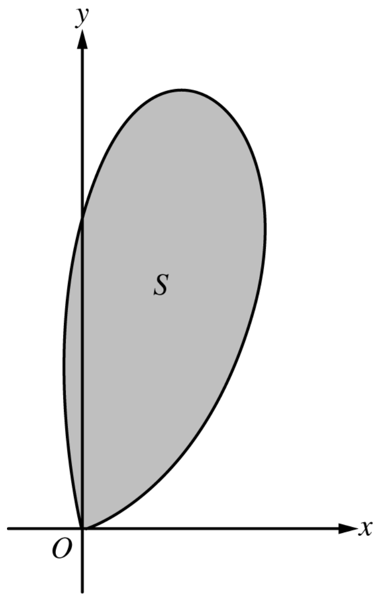
1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.



2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k\cos\theta$. Find $\lim_{k\to\infty} A(k)$.

END OF PART A OF SECTION II

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

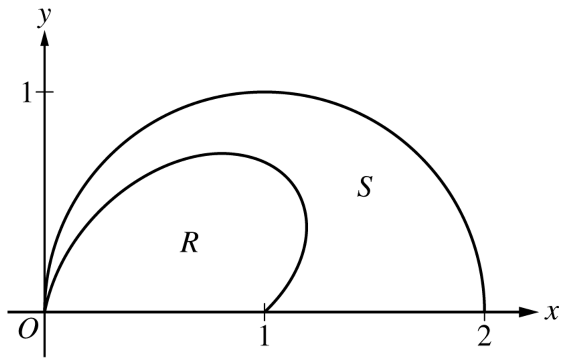
4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

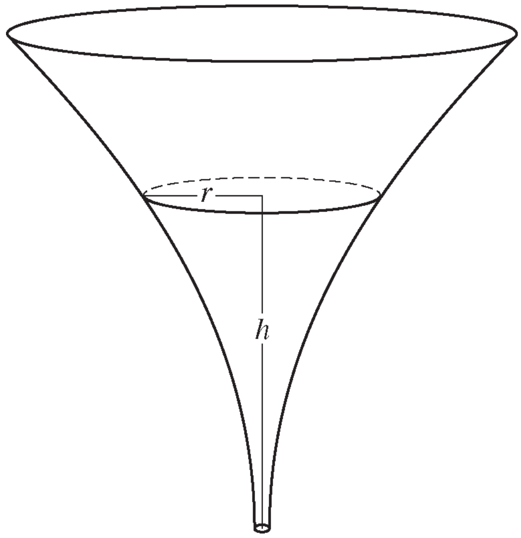
(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ? Give a reason for your answer.

END OF PART A OF SECTION II



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

   (a) Find the average value of the radius of the funnel.

   (b) Find the volume of the funnel.

   (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

8.2 Connecting Position, Velocity, and Acceleration Using Integrals

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS total distance 总路程 • displacement 位移

Objective

Build the skills to answer exam questions on **motion with integrals** — recovering velocity and position, and finding distance.

You must be able to:

- integrate acceleration to get velocity and velocity to get position
- find **displacement** 位移 $\int_a^b v\,dt$ and **total distance** 总路程 $\int_a^b |v|\,dt$
- use an initial condition to fix the constant

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Integrating the motion chain

Acceleration integrates to velocity; velocity integrates to position:

$$v(t) = \int a(t)\,dt, \qquad s(t) = \int v(t)\,dt.$$

Each integration adds a constant, fixed by an initial condition.

■ Displacement

Displacement over $[a, b]$ is the net change in position:

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

It can be negative (ended left of the start).

■ **Total distance**

Total distance counts all motion regardless of direction:

$$\int_a^b |v(t)|\,dt.$$

Split the integral where v changes sign, integrate each piece, and add the **absolute** amounts.

■ **A worked distance**

$v(t) = t - 2$ on $[0,3]$: $v < 0$ on $[0,2)$, $v > 0$ on $(2,3]$. Distance $= \left|\int_0^2 (t-2)\,dt\right| + \int_2^3 (t-2)\,dt = |-2| + \frac{1}{2} = 2.5$.

2 Practice

Now apply the methods above.

2.1 A particle has velocity $v(t) = 6t$. Find its displacement over $0 \le t \le 3$. [2]

2.2 With $a(t) = 4$ and $v(0) = 1$, find $v(t)$. [2]

2.3 A particle has $v(t) = 2t - 6$. State the time it changes direction. [1]

3 Exam-style questions

3.1 The total distance travelled over $[a,b]$ is [1]

- **A** $\displaystyle\int_a^b v\,dt$
- **B** $\displaystyle\int_a^b |v|\,dt$
- **C** $v(b) - v(a)$
- **D** $\displaystyle\int_a^b a\,dt$

3.2 A particle moves with velocity $v(t) = 3t^2 - 12$ m/s.

(a) Find the displacement over $0 \le t \le 3$. [2]

(b) At what time does the particle change direction? [2]

3.3 A particle has velocity $v(t) = t - 3$ on $0 \le t \le 5$.

(a) Find the displacement. [2]

(b) Find the total distance travelled. [3]

Solutions

2.1 $\displaystyle\int_0^3 6t\,dt = \left[3t^2\right]_0^3 = 27.$

2.2 $v(t) = 4t + C$; $v(0) = C = 1$, so $v(t) = 4t + 1$.

2.3 $v = 0$ at $2t - 6 = 0$, i.e. $t = 3$.

3.1 B — total distance is $\int |v|\,dt$.

3.2 (a) $\displaystyle\int_0^3 (3t^2 - 12)\,dt = \left[t^3 - 12t\right]_0^3 = 27 - 36 = -9$ m. (b) $v = 0$: $3t^2 = 12 \Rightarrow t = 2$.

3.3 (a) $\displaystyle\int_0^5 (t-3)\,dt = \left[\tfrac{t^2}{2} - 3t\right]_0^5 = 12.5 - 15 = -2.5$. (b) $v < 0$ on $[0,3)$, $v > 0$ on $(3,5]$; distance $= \left|\int_0^3\right| + \int_3^5 = |-4.5| + 2 = 6.5$.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

Objective

Build the skills to answer exam questions on **accumulation in applied contexts** — using a definite integral of a rate to find a total.

You must be able to:

- interpret $\int_a^b R(t)\,dt$ as the accumulated amount from a rate R
- combine a starting amount with $\int$(rate in − rate out)
- attach correct **units** and interpret in context

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Total from a rate

If a quantity changes at rate $R(t)$, the total change over $[a, b]$ is $\int_a^b R(t)\,dt$, carrying the units of $R\times$ time. Water flowing at $R(t) = 4t$ L/min adds $\int_0^3 4t\,dt = [2t^2]_0^3 = 18$ L over the first 3 minutes.

■ Starting amount plus accumulation

The amount at time b is

$$(\text{start}) + \int_a^b R(t)\,dt.$$

A tank with 50 L that gains at $R(t) = 4t$ over $[0, 3]$ holds $50 + 18 = 68$ L at $t = 3$.

■ Rate in minus rate out

If two rates act, integrate their **difference**. Water in at $I(t)$ and out at $O(t)$: net change $= \int_a^b \big(I(t) - O(t)\big)\,dt$.

■ **Interpreting a value**

Always state what the number **means**: "$\int_0^3 R\,dt = 18$ litres of water were added in the first 3 minutes" — a number **with** units and meaning.

2 Practice

Now apply the methods above.

2.1 Water flows in at $R(t) = 6$ L/min for 10 minutes. How much enters? [1]

2.2 A tank starts with 20 L and gains water at $R(t) = 2t$ L/min. Find the volume added over $0 \leq t \leq 4$. [2]

2.3 For the tank in 2.2, find the total volume in the tank at $t = 4$. [1]

3 Exam-style questions

3.1 If $R(t)$ is a rate in litres per hour, then $\displaystyle\int_0^5 R(t)\,dt$ has units of [1]

- **A** litres per hour
- **B** litres
- **C** hours
- **D** litres per hour squared

3.2 A city uses electricity at a rate $P(t) = 40 + 10t$ megawatts for $0 \leq t \leq 6$ hours.

(a) Find the total energy used over the 6 hours. [3]

(b) State the units. [1]

3.3 A reservoir holds 500 m^3. Water enters at $I(t) = 30$ m^3/h and leaves at $O(t) = 6t$ m^3/h, for $0 \le t \le 5$.

(a) Write an expression for the amount of water at time $t = 5$. [2]

(b) Evaluate it. [3]

Solutions

2.1 $6 \times 10 = 60$ L.

2.2 $\int_0^4 2t\,dt = [t^2]_0^4 = 16$ L.

2.3 $20 + 16 = 36$ L.

3.1 **B** — (litres/hour) $\times$ hours $=$ litres.

3.2 (a) $\int_0^6 (40 + 10t)\,dt = \left[40t + 5t^2\right]_0^6 = 240 + 180 = 420$. (b) megawatt-hours.

3.3 (a) $500 + \int_0^5 (30 - 6t)\,dt$. (b) $\left[30t - 3t^2\right]_0^5 = 150 - 75 = 75$; total $500 + 75 = 575$ m^3.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\dfrac{t}{100}\right)^3\left(1 - \dfrac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \le h \le 30$ and is modeled by $f(h)$ for $h \ge 30$. The continuous function f is not explicitly given.

(a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

(b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

(c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\, dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

(d) The boat is moving on the surface of the sea. At time $t \ge 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \le t \le 1$.

END OF PART A OF SECTION II

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

   (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

   (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

   (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

   (d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \le t \le 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

   (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \le t \le 8$?

   (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

   (c) At what time t, $0 \le t \le 8$, is the amount of water in the pipe at a minimum? Justify your answer.

   (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

8.4 Finding the Area Between Curves Expressed as Functions of x

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on the **area between two curves** (integrating in x).

You must be able to:

- find the **intersection points** that give the limits of integration
- integrate **top minus bottom**: $\int_a^b \left(f_{\text{top}} - f_{\text{bottom}}\right)dx$
- set up the area correctly even when a curve dips below the other

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Top minus bottom

The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) from a to b is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

■ Finding the limits

Set the curves equal to find where they cross. For $y = x$ and $y = x^2$: $x = x^2 \Rightarrow x = 0, 1$. These are the limits.

■ Which is on top?

Test a point between the crossings. At $x = 0.5$: the line $y = 0.5$ is above the parabola $y = 0.25$, so $y = x$ is the **top**:

$$\int_0^1 (x - x^2)\,dx = \left[\tfrac{x^2}{2} - \tfrac{x^3}{3}\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}.$$

■ **A worked area**

Area between $y = 4 - x^2$ and the x-axis ($y = 0$): crossings at $x = \pm 2$, so $\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$.

2 Practice

Now apply the methods above.

2.1 Find where $y = x^2$ meets $y = 4$. [1]

2.2 Set up (do not evaluate) the integral for the area between $y = x^2$ and $y = 4$. [2]

2.3 Evaluate $\int_0^2 (2x - x^2)\,dx$. [2]

3 Exam-style questions

3.1 The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) on $[a, b]$ is [1]

- **A** $\int_a^b (g - f)\,dx$
- **B** $\int_a^b (f - g)\,dx$
- **C** $\int_a^b f\,dx$
- **D** $\int_a^b (f + g)\,dx$

3.2 Find the area of the region enclosed by $y = x$ and $y = x^2$.

(a) Find the limits of integration. [1]

(b) Evaluate the area. [3]

3.3 Find the area of the region enclosed by $y = 6x - x^2$ and $y = x$. [5]

Solutions

2.1 $x^2 = 4 \Rightarrow x = \pm 2.$

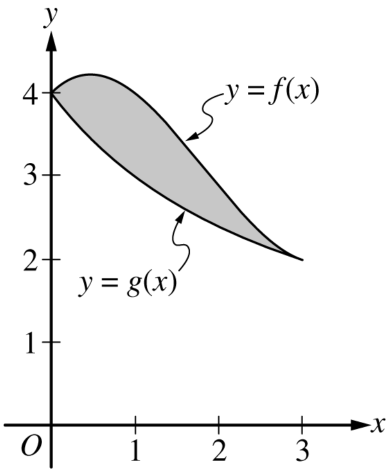
2.2 $\displaystyle\int_{-2}^{2} (4 - x^2)\, dx.$

2.3 $\left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$

3.1 **B** — integrate top minus bottom.

3.2 (a) $x = 0, 1.$ (b) $\displaystyle\int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$

3.3 Intersect: $6x - x^2 = x \Rightarrow x^2 - 5x = 0 \Rightarrow x = 0, 5$; top is $6x - x^2$; $\displaystyle\int_0^5 \big((6x - x^2) - x\big)\, dx = \int_0^5 (5x - x^2)\, dx = \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = \frac{125}{2} - \frac{125}{3} = \frac{125}{6} \approx 20.8.$



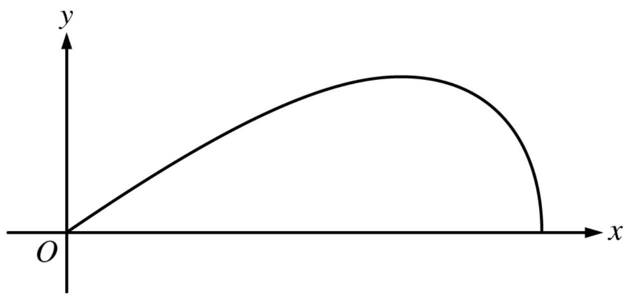
5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



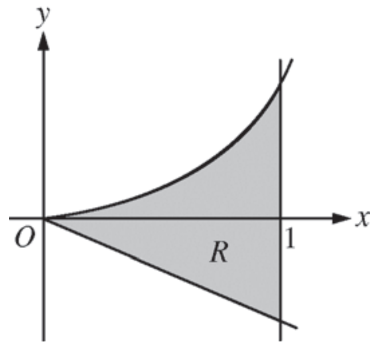
A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of R.

   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

8.5 Finding the Area Between Curves Expressed as Functions of y

Name: ________ Class: ________ Date: ________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on the **area between curves integrating in** y — when the curves are functions of y.

You must be able to:

- integrate **right minus left**: $\int_c^d (x_{\text{right}} - x_{\text{left}})\, dy$
- rewrite curves as $x = f(y)$
- choose dy to avoid splitting a region

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Right minus left

When a region is bounded left and right by curves, integrate in y:

$$\int_c^d \big(x_{\text{right}}(y) - x_{\text{left}}(y)\big)\, dy,$$

where c and d are the lower and upper y-limits.

Rewriting as x of y

$y = \sqrt{x}$ becomes $x = y^2$; $y = x$ stays $x = y$. For the region between $y = \sqrt{x}$ and $y = x$, in y: right curve $x = y$? Test $y = 0.5$: $y^2 = 0.25$ (left), $y = 0.5$ (right), so right $= y$, left $= y^2$.

Setting up the integral

Crossings of $x = y$ and $x = y^2$: $y = y^2 \Rightarrow y = 0, 1$. Area

$$\int_0^1 (y - y^2)\, dy = \left[\tfrac{y^2}{2} - \tfrac{y^3}{3}\right]_0^1 = \tfrac{1}{6}.$$

■ **Why choose dy**

Integrating in y turns a sideways-opening region (like one bounded by $x = y^2$) into a **single** integral, instead of splitting a top-and-bottom setup.

2 Practice

Now apply the methods above.

2.1 Rewrite $y = \sqrt{x}$ as a function $x = f(y)$. [1]

2.2 Find where $x = y$ meets $x = y^2$. [1]

2.3 Evaluate $\int_0^2 (4 - y^2)\, dy$. [2]

3 Exam-style questions

3.1 To find an area by integrating in y, you integrate [1]

- **A** top minus bottom
- **B** right minus left
- **C** $f(x) - g(x)$
- **D** the average of the curves

3.2 A region is bounded by $x = y^2$ and $x = 2y$.

(a) Find the y-limits of integration. [2]

(b) Set up and evaluate the area, integrating in y. [4]

3.3 Find the area enclosed by $x = y^2 - 1$ and $x = 3$ (integrate in y). [4]

Solutions

2.1 $x = y^2$ (for $y \geq 0$).

2.2 $y = y^2 \Rightarrow y = 0, 1$.

2.3 $\left[4y - \frac{y^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$.

3.1 **B** — integrating in y uses right minus left.

3.2 (a) $y^2 = 2y \Rightarrow y = 0, 2$. (b) right $= 2y$, left $= y^2$; $\displaystyle\int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.

3.3 $y^2 - 1 = 3 \Rightarrow y = \pm 2$; right $= 3$, left $= y^2 - 1$; $\displaystyle\int_{-2}^2 \big(3 - (y^2 - 1)\big)\, dy = \int_{-2}^2 (4 - y^2)\, dy = \left[4y - \frac{y^3}{3}\right]_{-2}^2 = \frac{32}{3}$.

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8.6 Finding the Area Between Curves That Intersect More Than Twice

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

Objective

Build the skills to answer exam questions on **area when curves cross more than twice** — splitting the region where top and bottom switch.

You must be able to:

- find **all** intersection points
- decide which curve is on top on **each** subinterval
- add the pieces (each set up as top minus bottom)

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ When top and bottom swap

If two curves cross more than twice, the upper curve **changes** between crossings. You must split the integral at each crossing and use the correct top minus bottom on each piece.

■ Finding all crossings

For $y = x^3$ and $y = x$: $x^3 = x \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = -1, 0, 1$. Three crossings, so **two** subintervals: $[-1, 0]$ and $[0, 1]$.

■ Deciding top on each piece

On $(0, 1)$ test $x = 0.5$: $x = 0.5$ vs $x^3 = 0.125$, so $y = x$ is on top. On $(-1, 0)$ test $x = -0.5$: $x^3 = -0.125$ vs $x = -0.5$, so $y = x^3$ is on top.

■ Adding the pieces

$$\text{Area} = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Using $\int |f - g|$ automatically handles the switch, but you must still split to integrate.

2 Practice

Now apply the methods above.

2.1 Find all solutions of $x^3 = 4x$. [2]

2.2 On $(0, 2)$, which is greater: $y = 4x$ or $y = x^3$? (test $x = 1$) [1]

2.3 Evaluate $\int_0^2 (4x - x^3)\,dx$. [2]

3 Exam-style questions

3.1 When two curves cross three times, the total enclosed area needs [1]

- **A** one integral
- **B** the integral split into two pieces with the correct top each time
- **C** the average of the curves
- **D** integration in y only

3.2 Find the total area enclosed between $y = x^3$ and $y = 4x$.

(a) Find all intersection points. [2]

(b) By symmetry or otherwise, evaluate the total enclosed area. [4]

3.3 The curves $y = x^3 - x$ and $y = 0$ enclose two regions. Explain why $\int_{-1}^{1} (x^3 - x)\, dx = 0$ does **not** give the total area, and describe the correct setup. [2]

Solutions

2.1 $x^3 - 4x = x(x^2-4) = 0 \Rightarrow x = 0, \pm 2$.

2.2 $4x = 4$ vs $x^3 = 1$ at $x = 1$, so $y = 4x$ is greater.

2.3 $\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8 - 4 = 4$.

3.1 **B** — split at the crossings, using the correct upper curve on each piece.

3.2 (a) $x^3 = 4x \Rightarrow x = -2, 0, 2$. (b) On $(0,2)$ top is $4x$, area $\int_0^2 (4x - x^3)\,dx = 4$; by symmetry the region on $(-2,0)$ has equal area 4; total $= 8$.

3.3 The integral is 0 because the region below the axis (negative signed area) cancels the region above; the true area is $\displaystyle\int_{-1}^{0} (x^3 - x)\,dx + \left|\int_0^1 (x^3 - x)\,dx\right|$, i.e. split at $x = 0$ and add the absolute areas.

8.7 Volumes with Cross Sections: Squares and Rectangles

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS cross sections 横截面 • cross-sectional area 横截面积

Objective

Build the skills to answer exam questions on **volumes by cross sections (squares and rectangles)** — integrating a slice's area.

You must be able to:

- write the **side length** of a slice as the gap between two curves
- integrate the **cross-sectional area** 横截面积: $V = \int_a^b A(x)\,dx$
- handle a square ($A = s^2$) or a rectangle cross section

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Volume as an integral of area

A solid whose cross section perpendicular to the x-axis has area $A(x)$ has volume

$$V = \int_a^b A(x)\,dx.$$

"Integrate the cross-sectional area" is the whole method.

■ Square cross sections

If each cross section is a **square** with side equal to the gap between $y = f(x)$ (top) and $y = g(x)$ (bottom), then $s = f(x) - g(x)$ and

$$V = \int_a^b \big(f(x) - g(x)\big)^2\,dx.$$

■ **A worked square-section volume**

Base bounded by $y = \sqrt{x}$ and the x-axis on $[0, 4]$, square cross sections: $s = \sqrt{x}$, so

$$V = \int_0^4 (\sqrt{x})^2 \, dx = \int_0^4 x \, dx = 8.$$

■ **Rectangle cross sections**

A rectangle of height $= k \times$ (base) has area $A = k\, s^2$ if height is proportional to the side; otherwise use the stated width and height. Always write $A(x)$ first, then integrate.

2 Practice

Now apply the methods above.

2.1 A slice has area $A(x) = x^2$. Write the volume integral over $[0, 3]$. [1]

2.2 A solid has square cross sections with side $s = 2x$ on $[0, 2]$. Find its volume. [3]

2.3 Evaluate $\int_0^1 (1 - x)^2 \, dx$. [2]

3 Exam-style questions

3.1 The volume of a solid with cross-sectional area $A(x)$ is [1]

- **A** $\int_a^b \sqrt{A(x)} \, dx$
- **B** $\int_a^b A(x) \, dx$
- **C** $\pi \int_a^b A(x) \, dx$

- **D** $\displaystyle\int_a^b A(x)^2\,dx$

3.2 The base of a solid is the region between $y = 4 - x^2$ and the x-axis. Cross sections perpendicular to the x-axis are squares.

(a) Write the side length s as a function of x. [1]

(b) Set up the volume integral (with limits). [2]

3.3 The base of a solid is bounded by $y = x$ and $y = x^2$. Cross sections perpendicular to the x-axis are squares with side equal to the gap between the curves. Set up and evaluate the volume. [5]

Solutions

2.1 $\displaystyle\int_0^3 x^2\,dx$.

2.2 $A = (2x)^2 = 4x^2$; $\displaystyle\int_0^2 4x^2\,dx = \left[\tfrac{4x^3}{3}\right]_0^2 = \tfrac{32}{3}$.

2.3 $\displaystyle\int_0^1 (1-x)^2\,dx = \left[-\tfrac{(1-x)^3}{3}\right]_0^1 = 0 - (-\tfrac{1}{3}) = \tfrac{1}{3}$.

3.1 **B** — integrate the cross-sectional area.

3.2 (a) $s = 4 - x^2$. (b) $\displaystyle\int_{-2}^2 (4-x^2)^2\,dx$.

3.3 $s = x - x^2$ on $[0,1]$; $V = \displaystyle\int_0^1 (x-x^2)^2\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \left[\tfrac{x^3}{3} - \tfrac{x^4}{2} + \tfrac{x^5}{5}\right]_0^1$
$= \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} = \tfrac{1}{30}$.

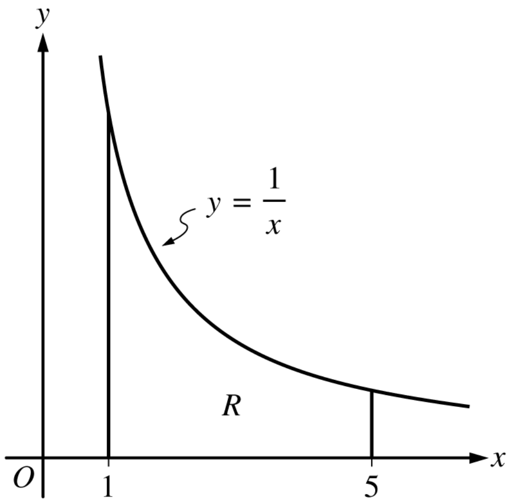


Figure 1

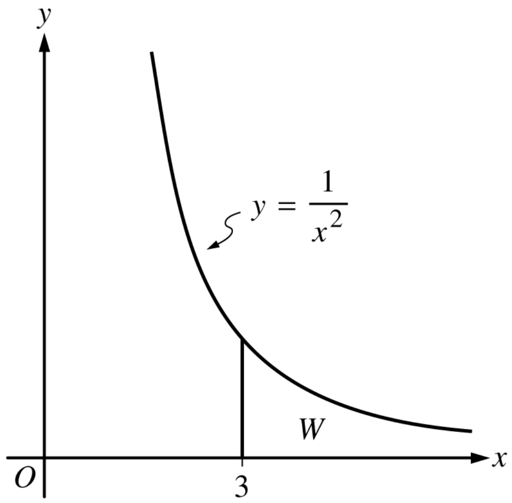


Figure 2

5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

   (a) Find the area of region R.

   (b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

   (c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

8.8 Volumes with Cross Sections: Triangles and Semicircles

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS equilateral triangle 等边三角形 • semicircular 半圆 • cross sections 横截面

Objective

Build the skills to answer exam questions on **volumes by cross sections (triangles and semicircles)** — using the right area formula for the slice.

You must be able to:

- use $A = \frac{\sqrt{3}}{4}s^2$ for an **equilateral triangle 等边三角形** slice
- use $A = \frac{\pi}{8}s^2$ for a **semicircular 半圆** slice (diameter $= s$)
- substitute the area formula and integrate

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Equilateral-triangle cross sections

A slice that is an equilateral triangle with side s has area $A = \frac{\sqrt{3}}{4}s^2$. If $s = f(x) - g(x)$, then

$$V = \int_a^b \frac{\sqrt{3}}{4}\big(f(x) - g(x)\big)^2\,dx.$$

■ Semicircular cross sections

A semicircle with **diameter** s has radius $\frac{s}{2}$ and area $\frac{1}{2}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2$. So

$$V = \int_a^b \frac{\pi}{8}\big(f(x) - g(x)\big)^2\,dx.$$

■ The method is always the same

Whatever the slice's shape, write its **area** in terms of the gap s, then integrate. Only the constant in front changes: 1 (square), $\frac{\sqrt{3}}{4}$ (triangle), $\frac{\pi}{8}$ (semicircle).

A worked triangle-section volume

Base bounded by $y = \sqrt{x}$ and the x-axis on $[0, 4]$, equilateral-triangle sections: $s = \sqrt{x}$, so $V = \int_0^4 \frac{\sqrt{3}}{4} x \, dx = \frac{\sqrt{3}}{4} \cdot 8 = 2\sqrt{3}$.

2 Practice

Now apply the methods above.

2.1 State the area of an equilateral triangle with side s. [1]

2.2 State the area of a semicircle with diameter s. [1]

2.3 A solid has equilateral-triangle cross sections with side $s = x$ on $[0, 2]$. Find its volume. [3]

3 Exam-style questions

3.1 A semicircular cross section with diameter s has area [1]

- **A** $\frac{\pi}{2}s^2$
- **B** $\frac{\pi}{4}s^2$
- **C** $\frac{\pi}{8}s^2$
- **D** πs^2

3.2 The base of a solid is bounded by $y = 2 - x$ and the axes in the first quadrant. Cross sections perpendicular to the x-axis are equilateral triangles.

(a) Write the side length s as a function of x. [1]

(b) Set up the volume integral. [2]

3.3 The base is the region between $y = \sqrt{x}$ and the x-axis on $[0, 4]$. Cross sections perpendicular to the x-axis are semicircles with diameter across the region. Set up and evaluate the volume. [4]

Solutions

2.1 $A = \frac{\sqrt{3}}{4}s^2$.

2.2 $A = \frac{\pi}{8}s^2$.

2.3 $V = \displaystyle\int_0^2 \frac{\sqrt{3}}{4}x^2\,dx = \frac{\sqrt{3}}{4}\left[\frac{x^3}{3}\right]_0^2 = \frac{\sqrt{3}}{4}\cdot\frac{8}{3} = \frac{2\sqrt{3}}{3}$.

3.1 **C** — radius $s/2$ gives $\frac{1}{2}\pi(s/2)^2 = \frac{\pi}{8}s^2$.

3.2 (a) $s = 2 - x$. (b) $\displaystyle\int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx$.

3.3 $s = \sqrt{x}$; $V = \displaystyle\int_0^4 \frac{\pi}{8}(\sqrt{x})^2\,dx = \frac{\pi}{8}\int_0^4 x\,dx = \frac{\pi}{8}\left[\frac{x^2}{2}\right]_0^4 = \frac{\pi}{8}\cdot 8 = \pi$.

8.9 Volume with Disc Method: Revolving Around the x- or y-Axis

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS disc method 圆盘法

Objective

Build the skills to answer exam questions on the **disc method** — volumes of revolution with no hole.

You must be able to:

- revolve a region around an axis and integrate $\pi(\text{radius})^2$ (the **disc method** 圆盘法)
- set the radius as the distance from the curve to the axis
- integrate in x (around the x-axis) or y (around the y-axis)

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The disc method

Revolving a region under $y = f(x)$ around the x-axis makes discs of radius $f(x)$:

$$V = \pi \int_a^b \big(f(x)\big)^2\,dx.$$

■ A worked disc volume

Revolve $y = \sqrt{x}$, $0 \le x \le 4$, around the x-axis: radius $\sqrt{x}$, so

$$V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi \cdot 8 = 8\pi.$$

■ Around the y-axis

Revolving around the y-axis uses $x = f(y)$ and integrates in y: $V = \pi \displaystyle\int_c^d \big(x(y)\big)^2\,dy$.

For $y = x^2$ (so $x = \sqrt{y}$) from $y = 0$ to 4: $V = \pi \int_0^4 (\sqrt{y})^2\,dy = \pi \int_0^4 y\,dy = 8\pi$.

■ **Radius is a distance**

The radius is always the **distance from the curve to the axis** of revolution — here just $f(x)$ or $x(y)$ because the axis is a coordinate axis.

2 Practice

Now apply the methods above.

2.1 Write the disc-method integral for revolving $y = x$ on $[0, 2]$ about the x-axis. [1]

2.2 Evaluate $\pi \displaystyle\int_0^2 x^2\, dx$. [2]

2.3 Find the volume when $y = \sqrt{x}$ on $[0, 9]$ is revolved about the x-axis. [3]

3 Exam-style questions

3.1 Revolving $y = f(x)$ about the x-axis gives the volume [1]

- **A** $\displaystyle\int_a^b f(x)\, dx$
- **B** $\pi \displaystyle\int_a^b f(x)\, dx$
- **C** $\pi \displaystyle\int_a^b \big(f(x)\big)^2\, dx$
- **D** $2\pi \displaystyle\int_a^b f(x)\, dx$

3.2 The region under $y = x^2$ from $x = 0$ to $x = 2$ is revolved about the x-axis.

(a) Write the volume integral. [2]

(b) Evaluate it. [3]

3.3 The region bounded by $x = y^2$, $y = 0$ to $y = 2$, and the y-axis is revolved about the y-axis. Find the volume. [4]

Solutions

2.1 $\pi \displaystyle\int_0^2 x^2\,dx$.

2.2 $\pi\left[\frac{x^3}{3}\right]_0^2 = \frac{8\pi}{3}$.

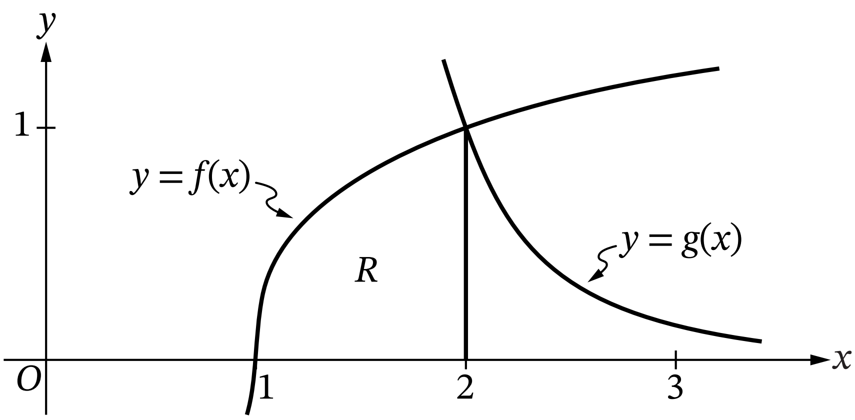
2.3 $\pi \displaystyle\int_0^9 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^9 = \frac{81\pi}{2}$.

3.1 **C** — the disc method integrates $\pi(\text{radius})^2$.

3.2 (a) $\pi \displaystyle\int_0^2 (x^2)^2\,dx = \pi \int_0^2 x^4\,dx$. (b) $\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$.

3.3 radius $= x = y^2$; $V = \pi \displaystyle\int_0^2 (y^2)^2\,dy = \pi \int_0^2 y^4\,dy = \pi\left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$.

5. Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2,1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)\,dx$. Show the work that leads to your answer.

Part B

Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

Part D

Evaluate $\int_2^{\infty} g(x)\,dx$. Show the work that leads to your answer.

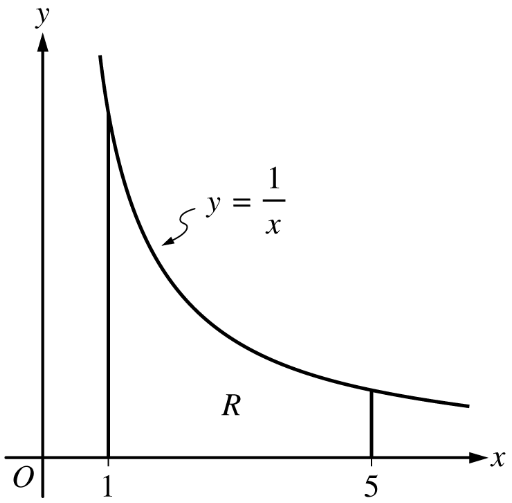


Figure 1

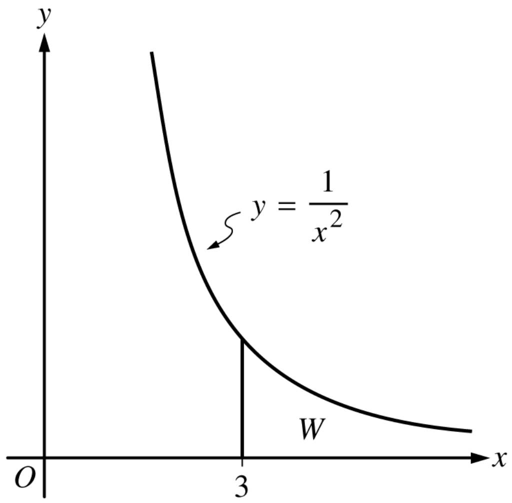


Figure 2

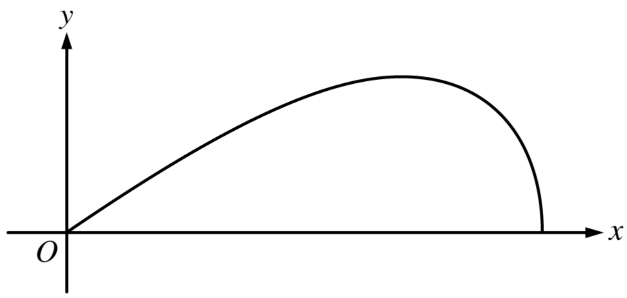
5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



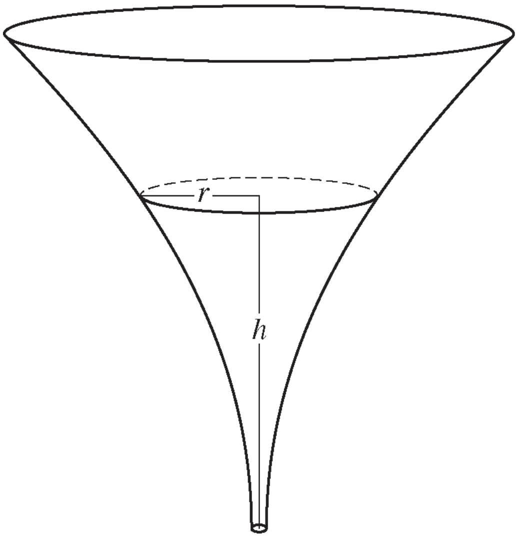
. A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

   (a) Find the average value of the radius of the funnel.

   (b) Find the volume of the funnel.

   (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

8.10 Volume with Disc Method: Revolving Around Other Axes

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS disc method 圆盘法

Objective

Build the skills to answer exam questions on the **disc method around other axes** — when the axis is a line like $y = k$.

You must be able to:

- adjust the **radius** to the distance from the curve to a non-coordinate axis
- write $R = |f(x) - k|$ for a horizontal axis $y = k$ (or $|x(y) - k|$ for $x = k$)
- integrate πR^2 as before

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Radius to a shifted axis

When the axis of revolution is $y = k$ (not the x-axis), the radius is the **distance from the curve to that line**:

$$R = |f(x) - k|, \qquad V = \pi \int_a^b \big(f(x) - k\big)^2 \, dx.$$

A worked example

Revolve $y = \sqrt{x}$, $0 \le x \le 4$, about the line $y = -1$. The radius is $\sqrt{x} - (-1) = \sqrt{x} + 1$:

$$V = \pi \int_0^4 (\sqrt{x} + 1)^2 \, dx = \pi \int_0^4 (x + 2\sqrt{x} + 1) \, dx.$$

Revolving about a vertical line

For an axis $x = k$, use $R = |x(y) - k|$ and integrate in y. Always draw a picture and measure the radius from the curve **to** the axis.

■ **Watch the direction**

If the region lies **below** the axis line, the distance is $k - f(x)$; squaring makes the sign irrelevant, but set the radius as a genuine (positive) distance.

2 Practice

Now apply the methods above.

2.1 A curve $y = f(x)$ is revolved about $y = 3$. Write the radius. [1]

2.2 Revolving $y = 2$ (constant) region about $y = -1$: state the radius. [1]

2.3 Expand $(\sqrt{x} + 1)^2$. [1]

3 Exam-style questions

3.1 Revolving $y = f(x)$ about the line $y = k$ uses radius [1]

- **A** $f(x)$
- **B** $f(x) - k$
- **C** k
- **D** $f(x) + k$ always

3.2 The region under $y = x^2$ from $x = 0$ to $x = 2$ is revolved about the line $y = -1$.

(a) Write the radius $R(x)$. [1]

(b) Set up the volume integral. [2]

3.3 The region under $y = \sqrt{x}$ on $[0, 4]$ is revolved about the line $y = 2$. Set up the volume integral for the solid (radius $= 2 - \sqrt{x}$). [3]

Solutions

2.1 $R = |f(x) - 3|$.

2.2 $R = 2 - (-1) = 3$.

2.3 $x + 2\sqrt{x} + 1$.

3.1 **B** — the radius is the distance $f(x) - k$ to the line.

3.2 (a) $R = x^2 - (-1) = x^2 + 1$. (b) $\pi \int_0^2 (x^2 + 1)^2 \, dx$.

3.3 $R = 2 - \sqrt{x}$; $V = \pi \int_0^4 (2 - \sqrt{x})^2 \, dx$.

AP CALCULUS BC • EXERCISE SHEET

8.11 Volume with Washer Method: Revolving Around the x- or y-Axis

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS washer method 垫圈法

Objective

Build the skills to answer exam questions on the **washer method** — volumes of revolution with a hole.

You must be able to:

- identify the **outer** and **inner** radii of a washer (ring)
- integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$ (the **washer method 垫圈法**)
- set each radius as a distance from a curve to the axis

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

When there is a hole

If the region does **not** touch the axis, revolving it leaves a hole, so each slice is a **washer** (a ring). Its area is (outer disc) − (inner disc):

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx.$$

Identifying the radii

Revolve the region between $y = \sqrt{x}$ (top) and $y = x$ (bottom) on $[0, 1]$ about the x-axis. Outer radius (further curve) $R = \sqrt{x}$; inner radius $r = x$:

$$V = \pi \int_0^1 \left((\sqrt{x})^2 - x^2\right) dx = \pi \int_0^1 (x - x^2)\, dx = \pi\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{\pi}{6}.$$

Do not square then subtract wrongly

$R^2 - r^2$ is **not** $(R - r)^2$. Square each radius separately, then subtract.

■ **Choosing outer vs inner**

The **outer** radius comes from the curve **farther** from the axis; the **inner** from the curve **nearer** the axis. A quick sketch settles which is which.

2 Practice

Now apply the methods above.

2.1 State the washer-method formula for the volume. [1]

2.2 For $R = 4$ and $r = 2$, find $\pi(R^2 - r^2)$. [1]

2.3 Evaluate $\pi \int_0^1 (x - x^2)\,dx$. [2]

3 Exam-style questions

3.1 The washer method integrates [1]

- **A** $\pi \int (R - r)^2\,dx$
- **B** $\pi \int (R^2 - r^2)\,dx$
- **C** $\pi \int R^2\,dx$
- **D** $\int (R^2 - r^2)\,dx$

3.2 The region between $y = x$ and $y = x^2$ (from $x = 0$ to $x = 1$) is revolved about the x-axis.

(a) Identify the outer and inner radii. [2]

(b) Set up and evaluate the volume. [4]

3.3 The region bounded by $y = 4$ and $y = x^2$ (for $0 \leq x \leq 2$) is revolved about the x-axis. Set up the washer integral for the volume. [3]

Solutions

2.1 $V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx$.

2.2 $\pi(16 - 4) = 12\pi$.

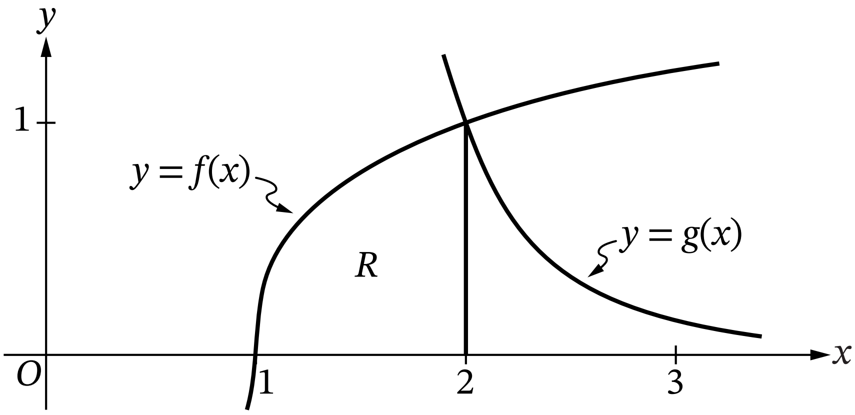
2.3 $\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$.

3.1 **B** — subtract the **squares** of the radii.

3.2 (a) On $(0, 1)$, $y = x$ is above $y = x^2$; outer $R = x$, inner $r = x^2$. (b) $V = \pi \int_0^1 (x^2 - x^4)\, dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$.

3.3 Outer $R = 4$, inner $r = x^2$; $V = \pi \int_0^2 \left(4^2 - (x^2)^2\right) dx = \pi \int_0^2 (16 - x^4)\, dx$.

5. Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)dx$. Show the work that leads to your answer.

Part B

Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

Part D

Evaluate $\int_2^{\infty} g(x)dx$. Show the work that leads to your answer.

8.12 Volume with Washer Method: Revolving Around Other Axes

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS washer method 垫圈法

Objective

Build the skills to answer exam questions on the **washer method around other axes** — a hole plus a shifted axis.

You must be able to:

- shift **both** radii to the distance from each curve to a line $y = k$ or $x = k$
- integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$ with the shifted radii
- sketch the region and label the two radii

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Both radii shift

Revolving about a line $y = k$ moves **both** radii. Each becomes the distance from its curve to that line:

$$R_{\text{outer}} = |f_{\text{far}} - k|, \quad R_{\text{inner}} = |f_{\text{near}} - k|.$$

■ A worked example

Revolve the region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ about the line $y = -1$. Distances to $y = -1$: outer (from $\sqrt{x}$) $= \sqrt{x} + 1$; inner (from x) $= x + 1$:

$$V = \pi \int_0^1 \left((\sqrt{x} + 1)^2 - (x + 1)^2\right) dx.$$

■ Keep the order

The outer curve (farther from the axis) still gives the larger radius. When the axis is **below** the region, adding the same amount to each keeps their order.

■ **Method summary**

Sketch the region and the axis, mark R_{outer} and R_{inner} as distances to the axis, then integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$.

2 Practice

Now apply the methods above.

2.1 A curve $y = f(x)$ is revolved about $y = -2$. Write its radius. [1]

2.2 For outer curve $y = 4$ and inner curve $y = 1$ revolved about $y = -1$, state both radii. [2]

2.3 Expand $(x+1)^2 - (\sqrt{x}+1)^2$? No — first expand $(x+1)^2$. [1]

3 Exam-style questions

3.1 Revolving a region between two curves about $y = k$ uses radii [1]

- **A** the curves themselves
- **B** each curve's distance to the line $y = k$
- **C** only the outer curve
- **D** k for both

3.2 The region between $y = x^2$ (lower) and $y = 2x$ (upper) on $[0, 2]$ is revolved about the line $y = -1$.

(a) Write the outer and inner radii. [2]

(b) Set up the volume integral. [2]

3.3 The region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ is revolved about the line $y = 2$. Write the outer and inner radii (as distances to $y = 2$), and set up the volume integral. [4]

Solutions

2.1 $R = f(x) + 2$.

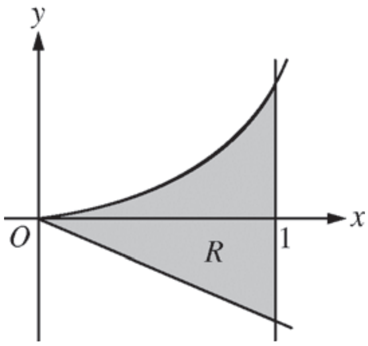
2.2 outer $= 4 - (-1) = 5$; inner $= 1 - (-1) = 2$.

2.3 $(x + 1)^2 = x^2 + 2x + 1$.

3.1 B — each radius is the curve's distance to the line $y = k$.

3.2 (a) On $(0, 2)$, $2x \geq x^2$, so outer (from $2x$) $= 2x + 1$, inner (from x^2) $= x^2 + 1$. (b) $V = \pi \int_0^2 \left((2x + 1)^2 - (x^2 + 1)^2\right) dx$.

3.3 Distances to $y = 2$: outer (from $y = x$, the lower curve, farther below 2) $= 2 - x$; inner (from $y = \sqrt{x}$) $= 2 - \sqrt{x}$; $V = \pi \int_0^1 \left((2 - x)^2 - (2 - \sqrt{x})^2\right) dx$.



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of R.

   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

AP CALCULUS BC • EXERCISE SHEET

8.13 The Arc Length of a Smooth Curve and Distance Traveled

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS arc length 弧长

Objective

Build the skills to answer exam questions on **arc length** — the length of a smooth curve.

You must be able to:

- apply $L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$
- set up the integral from a given function
- recognise the same idea as **distance travelled** along a path

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The arc-length formula

The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

It adds up tiny hypotenuses $\sqrt{dx^2 + dy^2}$ along the curve.

■ Setting up the integral

For $y = \frac{2}{3}x^{3/2}$: $f'(x) = x^{1/2}$, so $\left(f'\right)^2 = x$, and

$$L = \int_0^3 \sqrt{1 + x}\, dx.$$

■ Evaluating

$$\int_0^3 \sqrt{1 + x}\, dx = \left[\tfrac{2}{3}(1 + x)^{3/2}\right]_0^3 = \tfrac{2}{3}\left(8 - 1\right) = \tfrac{14}{3}.$$

■ **Distance along a path**

If a particle moves along $y = f(x)$, the **distance travelled** is the same arc-length integral. Many arc-length integrals must be evaluated numerically — set them up correctly even when you cannot integrate by hand.

2 Practice

Now apply the methods above.

2.1 State the arc-length formula for $y = f(x)$ on $[a, b]$. [1]

2.2 For $y = \frac{2}{3}x^{3/2}$, find $f'(x)$ and $\left(f'(x)\right)^2$. [2]

2.3 Set up (do not evaluate) the arc length of $y = x^2$ from $x = 0$ to $x = 2$. [2]

3 Exam-style questions

3.1 The arc length of $y = f(x)$ on $[a, b]$ is [1]

- **A** $\int_a^b f(x)\,dx$
- **B** $\int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$
- **C** $\int_a^b f'(x)\,dx$
- **D** $\pi \int_a^b \left(f(x)\right)^2\,dx$

3.2 A curve is $y = \frac{2}{3}x^{3/2}$ for $0 \le x \le 3$.

(a) Set up the arc-length integral. [2]

(b) Evaluate it. [3]

3.3 Set up the integral for the length of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$. (Use $\frac{d}{dx}\ln(\cos x) = -\tan x$ and $1 + \tan^2 x = \sec^2 x$.) [3]

Solutions

2.1 $L = \displaystyle\int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx.$

2.2 $f'(x) = x^{1/2}$; $\big(f'(x)\big)^2 = x.$

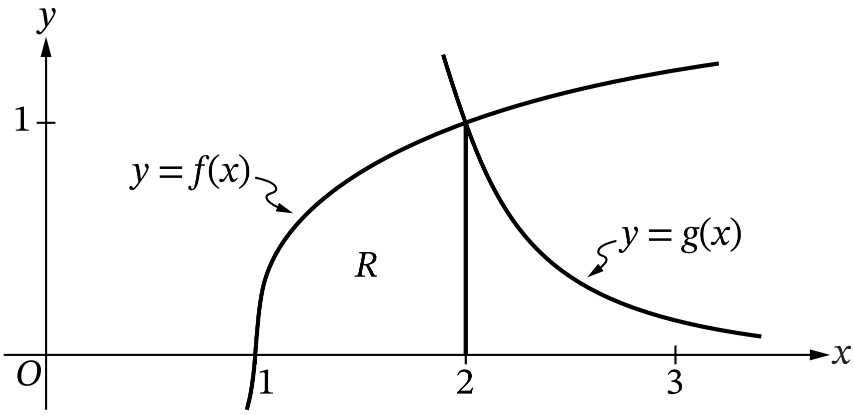
2.3 $f'(x) = 2x$, so $L = \displaystyle\int_0^2 \sqrt{1 + 4x^2}\, dx.$

3.1 **B** — arc length integrates $\sqrt{1 + (f')^2}$.

3.2 (a) $f'(x) = x^{1/2}$, so $L = \displaystyle\int_0^3 \sqrt{1 + x}\, dx$. (b) $\left[\frac{2}{3}(1+x)^{3/2}\right]_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$.

3.3 $f'(x) = -\tan x$, so $1 + (f')^2 = 1 + \tan^2 x = \sec^2 x$; $L = \displaystyle\int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx.$

5. Let *f* be the function defined by $f(x) = \sqrt[3]{x-1}$, and let *g* be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of *f* and *g* intersect at the point $(2, 1)$. Let *R* be the region bounded by the graph of *f*, the *x*-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)dx$. Show the work that leads to your answer.

Part B

Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region *R* is rotated about the *x*-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region *R*.

Part D

Evaluate $\int_2^{\infty} g(x)dx$. Show the work that leads to your answer.

x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^{\pi} \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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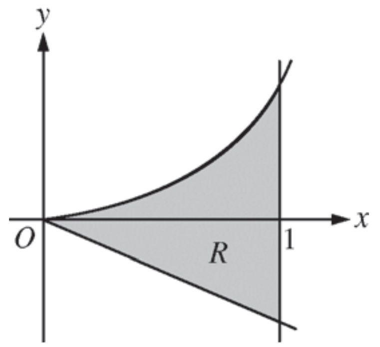

5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.