Week 17

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

AP Calculus BC Name: ____________________ Score: ______

1. A differential equation relates a function to its...
   - ☐ inverse
   - ☐ derivatives
   - ☐ reciprocal
   - ☐ domain

2. To verify a candidate solution, you...
   - ☐ integrate both sides
   - ☐ differentiate it and substitute into the equation
   - ☐ plug in one value of x
   - ☐ graph it

3. For $\dfrac{dy}{dx} = x + y$, find the slope of the dash at the point $(2, 3)$.

   ____

4. A solution curve through the slope field must always stay...
   - ☐ perpendicular to the dashes
   - ☐ tangent to the dashes
   - ☐ horizontal
   - ☐ a straight line

5. Euler's update rule is $y_{n+1} =$
   - ☐ $y_n \cdot h\, f(x_n, y_n)$
   - ☐ $y_n + h\, f(x_n, y_n)$
   - ☐ $y_n + f(x_n, y_n)$
   - ☐ $h\, f(x_n, y_n)$

6. Which equation is separable?
   - ☐ $\dfrac{dy}{dx} = x + y$
   - ☐ $\dfrac{dy}{dx} = xy$
   - ☐ $\dfrac{dy}{dx} = x^2 + y^2$

☐ $\dfrac{dy}{dx} = \sin(x+y)$

7. $y = e^{3x}$ is a solution of $\dfrac{dy}{dx} = 3y$.

   ☐ True ☐ False

8. Two different solution curves of the same equation can cross each other.

   ☐ True ☐ False

9. For $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, $h = 0.5$: the slope at $(0, 1)$ is 1. Find y_1.

   ____

10. For $y = x^2 + C$ with $y(1) = 5$, find C.

   ____

AP Calculus BC

1. **derivatives**

   It involves the function and its derivative(s).

2. **differentiate it and substitute into the equation**

   Differentiate the candidate, substitute, check both sides match.

3. **5**

   $2 + 3 = 5$.

4. **tangent to the dashes**

   It follows the tangent directions.

5. $y_n + h\, f(x_n, y_n)$

   Old y plus slope times step.

6. $\dfrac{dy}{dx} = xy$

   xy factors as (x-part)×(y-part); a sum does not.

7. **True**

   $\frac{dy}{dx} = 3e^{3x} = 3y$. ✓

8. **False**

   Each point has one slope, so solution curves never cross.

9. **1.5**

   $1 + 0.5(1) = 1.5$.

10. **4**

    $5 = 1 + C \Rightarrow C = 4$.

AP CALCULUS BC • EXERCISE SHEET

7.1 Modeling Situations with Differential Equations

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS differential equation 微分方程

Objective

Build the skills to answer exam questions on **modeling with differential equations** — turning a description of a rate into an equation.

You must be able to:

- write a **differential equation 微分方程** from a verbal statement about a rate
- recognise "proportional to" as multiplication by a constant k
- identify the quantity, its rate $\frac{dy}{dt}$, and what it depends on

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ "Rate of change" is a derivative

"The population P changes over time t" means the rate is $\frac{dP}{dt}$. The rest of the sentence tells you what it equals.

■ "Proportional to"

"Proportional to" means "equal to a constant times". "The rate of growth is proportional to the population" becomes

$$\frac{dP}{dt} = kP.$$

■ Proportional to a difference

Newton's law of cooling: an object's temperature T changes at a rate proportional to the difference between T and the room temperature T_r:

$$\frac{dT}{dt} = k(T - T_r).$$

■ **Reading the sign of k**

A **growing** quantity has $k > 0$; a **decaying** or cooling one has $k < 0$. The differential equation captures the model before you ever solve it.

2 Practice

Now apply the methods above.

2.1 Write a differential equation for: "the amount A decreases at a rate proportional to A." [2]

2.2 A tank drains so that its volume V falls at a constant 3 litres per minute. Write the differential equation. [1]

2.3 Write a differential equation for: "the number of bacteria N grows at a rate proportional to N." [1]

3 Exam-style questions

3.1 "The rate of change of y is proportional to y" is written [1]

- **A** $y = kt$
- **B** $\dfrac{dy}{dt} = k$
- **C** $\dfrac{dy}{dt} = ky$
- **D** $\dfrac{dy}{dt} = k + y$

3.2 A cup of coffee at temperature T cools in a room at 20°C. Its temperature falls at a rate proportional to how much hotter it is than the room.

(a) Write a differential equation for $\dfrac{dT}{dt}$. [2]

(b) State the sign of the constant of proportionality, with a reason. [1]

3.3 A savings account earns interest continuously at 4% per year, so the balance B grows at a rate of $0.04B$ per year, while \$2000 per year is also added. Write a differential equation for $\dfrac{dB}{dt}$. [2]

Solutions

2.1 $\dfrac{dA}{dt} = kA$ with $k < 0$ (or $\dfrac{dA}{dt} = -kA$, $k > 0$).

2.2 $\dfrac{dV}{dt} = -3$.

2.3 $\dfrac{dN}{dt} = kN$.

3.1 **C** — proportional to y means $\dfrac{dy}{dt} = ky$.

3.2 (a) $\dfrac{dT}{dt} = k(T - 20)$. (b) $k < 0$ — the coffee is cooling, so its temperature decreases.

3.3 $\dfrac{dB}{dt} = 0.04B + 2000$.

7.2 Verifying Solutions for Differential Equations

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS initial condition 初始条件 • differential equation 微分方程

Objective

Build the skills to answer exam questions on **verifying solutions of differential equations** — checking that a function satisfies a given equation.

You must be able to:

- differentiate a proposed solution and **substitute** it into the equation
- confirm both sides are equal for all x (or t)
- check that an **initial condition** 初始条件 is also met

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Substitute and compare

To verify $y = e^{2x}$ solves $\dfrac{dy}{dx} = 2y$: differentiate, $\dfrac{dy}{dx} = 2e^{2x}$; the right side is $2y = 2e^{2x}$. They match, so $y = e^{2x}$ **is** a solution.

A solution with a constant

$y = Ce^{kt}$ solves $\dfrac{dy}{dt} = ky$ for **any** constant C: $\dfrac{dy}{dt} = kCe^{kt} = ky$. This is the general solution; each C gives one curve.

Checking an implicit solution

To check $y^2 = x^2 + C$ solves $\dfrac{dy}{dx} = \dfrac{x}{y}$: differentiate implicitly, $2y\dfrac{dy}{dx} = 2x$, so $\dfrac{dy}{dx} = \dfrac{x}{y}$. Verified.

Including an initial condition

If asked to verify a **particular** solution, also check the point. $y = 3e^{2x}$ with $y(0) = 3$: at $x = 0$, $y = 3e^0 = 3$. The condition holds.

2 Practice

Now apply the methods above.

2.1 Verify that $y = e^{5x}$ satisfies $\dfrac{dy}{dx} = 5y$. [2]

2.2 Show that $y = 4e^{-t}$ satisfies $\dfrac{dy}{dt} = -y$. [2]

2.3 For $y = Ce^{3x}$, state $\dfrac{dy}{dx}$ in terms of y. [1]

3 Exam-style questions

3.1 Which function satisfies $\dfrac{dy}{dx} = 2y$? [1]

- **A** $y = 2x$
- **B** $y = x^2$
- **C** $y = e^{2x}$
- **D** $y = 2e^x$

3.2 Consider $y = x^2 + 3x + C$.

(a) Show that $\dfrac{dy}{dx} = 2x + 3$. [1]

(b) Find the value of C so that the solution passes through $(1, 6)$. [2]

3.3 Verify that $y = \sin(2x)$ satisfies $\dfrac{d^2y}{dx^2} = -4y$. [3]

Solutions

2.1 $\dfrac{dy}{dx} = 5e^{5x}$; this equals $5y = 5e^{5x}$, so it is a solution.

2.2 $\dfrac{dy}{dt} = -4e^{-t}$; and $-y = -4e^{-t}$, so they match.

2.3 $\dfrac{dy}{dx} = 3Ce^{3x} = 3y$.

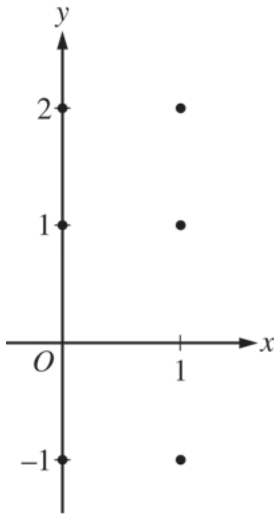
3.1 **C** — $\dfrac{d}{dx}e^{2x} = 2e^{2x} = 2y$.

3.2 (a) $\dfrac{dy}{dx} = 2x + 3$. (b) $6 = 1 + 3 + C \Rightarrow C = 2$.

3.3 $\dfrac{dy}{dx} = 2\cos(2x)$; $\dfrac{d^2y}{dx^2} = -4\sin(2x)$; this equals $-4y = -4\sin(2x)$.

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

7.3 Sketching Slope Fields

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS slope field 斜率场 • differential equation 微分方程

Objective

Build the skills to answer exam questions on **slope fields** — drawing and reading the little segments of $\dfrac{dy}{dx}$.

You must be able to:

- compute the slope $\dfrac{dy}{dx}$ at grid points and draw short segments (a **slope field** 斜率场)
- match a differential equation to its slope field
- sketch a solution curve that **follows** the segments through a given point

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Computing a slope at a point

For $\dfrac{dy}{dx} = x + y$, the slope at $(1, 2)$ is $1 + 2 = 3$: draw a short segment of slope 3 there. Repeating at many points gives the slope field.

■ Slopes that depend only on x

If $\dfrac{dy}{dx} = 2x$, the slope depends on x **only**, so every segment in a vertical column has the same slope — the field looks the same as you move up or down.

■ Slopes that depend only on y

If $\dfrac{dy}{dx} = y$, segments in a horizontal row match. Where $y = 0$ the slope is 0 (horizontal segments along the x-axis).

■ Following the field

A solution curve is tangent to the segments everywhere. Starting at a point, draw a curve that always runs **along** the nearby segments — like a boat following a current.

2 Practice

Now apply the methods above.

2.1 For $\dfrac{dy}{dx} = x - y$, find the slope at $(3, 1)$. [1]

2.2 For $\dfrac{dy}{dx} = xy$, find the slope at $(2, -1)$. [1]

2.3 For $\dfrac{dy}{dx} = y - 1$, state where all the segments are **horizontal**. [1]

3 Exam-style questions

3.1 In the slope field for $\dfrac{dy}{dx} = 2x$, along the line $x = 0$ the segments are [1]

- **A** vertical
- **B** horizontal
- **C** slope +1
- **D** slope +2

3.2 A slope field is given by $\dfrac{dy}{dx} = x + 1$.

(a) Find the slope at the points $(-1, 0)$, $(0, 2)$, and $(2, -3)$. [3]

(b) Along which vertical line are all segments horizontal? [1]

3.3 A slope field has horizontal segments only along the x-axis ($y = 0$) and gets steeper as $|y|$ grows, with the same pattern in every vertical column. Which differential equation fits: $\dfrac{dy}{dx} = x$, $\dfrac{dy}{dx} = y$, or $\dfrac{dy}{dx} = xy$? Explain. [2]

Solutions

2.1 $3 - 1 = 2$.

2.2 $(2)(-1) = -2$.

2.3 Where $y - 1 = 0$, i.e. along the line $y = 1$.

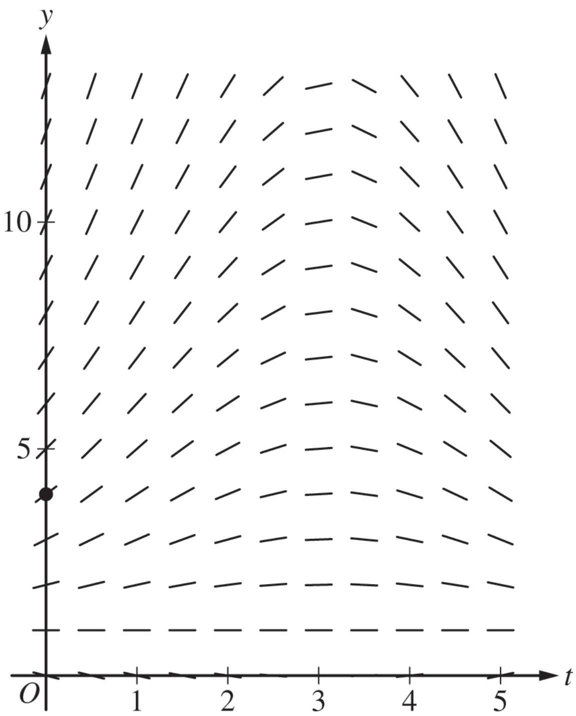
3.1 **B** — at $x = 0$ the slope is $2(0) = 0$, horizontal.

3.2 (a) $(-1, 0)$: 0; $(0, 2)$: 1; $(2, -3)$: 3. (b) $x = -1$ (where $x + 1 = 0$).

3.3 $\dfrac{dy}{dx} = y$ — the slope is 0 where $y = 0$ and grows with $|y|$, and depends on y only, so it is the same in every column.

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



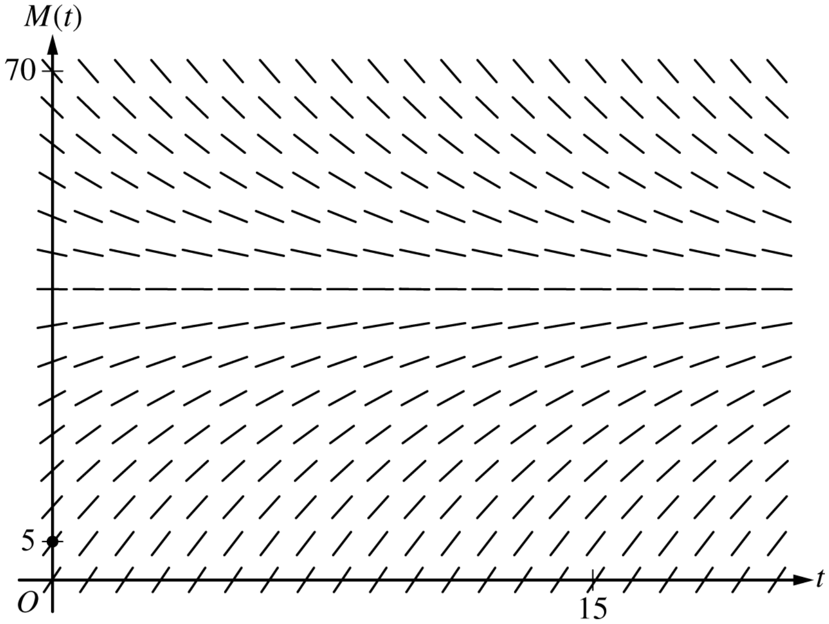
(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

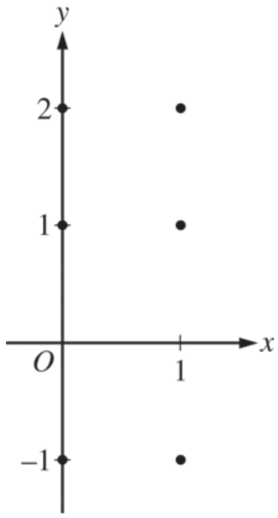
(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

7.4 Reasoning Using Slope Fields

Name: ____________ Class: ________ Date: __________ **Total: 11 marks**

KEY WORDS equilibrium solutions 平衡解 • differential equation 微分方程 • slope field 斜率场

Objective

Build the skills to answer exam questions on **reasoning with slope fields** — drawing conclusions about solutions without solving.

You must be able to:

- sketch the **particular solution** through a given point on a slope field
- describe **long-run behavior** (where solutions level off or grow)
- identify **equilibrium solutions** 平衡解 where $\dfrac{dy}{dx} = 0$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Equilibrium solutions

An **equilibrium** is a constant solution where $\dfrac{dy}{dx} = 0$ everywhere. For $\dfrac{dy}{dx} = y - 2$, setting $y - 2 = 0$ gives the equilibrium $y = 2$: a horizontal line the other solutions approach or leave.

■ Long-run behavior

For $\dfrac{dy}{dx} = y - 2$ with a start **below** $y = 2$: there $\dfrac{dy}{dx} < 0$, so the solution moves **away** downward. Starting above, it moves up. So $y = 2$ is an **unstable** equilibrium.

■ Sketching a particular solution

To sketch the solution through $(0, 1)$: from that point, follow the segments — the curve is always tangent to the local slope, curving as the field directs. Do not cross an equilibrium line.

■ Reading a maximum from the field

Where a solution curve has a **horizontal** tangent ($\dfrac{dy}{dx} = 0$) and the slope changes $+$ to $-$ around it, the solution has a local maximum there.

2 Practice

Now apply the methods above.

2.1 Find the equilibrium solution of $\frac{dy}{dx} = y - 5$. [1]

2.2 For $\frac{dy}{dx} = 3 - y$, state the equilibrium value of y. [1]

2.3 For $\frac{dy}{dx} = y - 5$, if a solution starts at $y = 6$, does it move toward or away from $y = 5$? [1]

3 Exam-style questions

3.1 An equilibrium solution of a differential equation is a value of y where [1]

- **A** $\frac{dy}{dx} = 0$
- **B** $y = 0$
- **C** $x = 0$
- **D** $\frac{dy}{dx}$ is undefined

3.2 Consider $\frac{dy}{dx} = 2 - y$.

(a) Find the equilibrium solution. [1]

(b) For a solution starting at $y = 5$, state whether y increases or decreases, with a reason. [2]

(c) State $\lim_{x\to\infty} y$ for that solution. [1]

3.3 A slope field for $\dfrac{dy}{dx} = y(4 - y)$ has two equilibrium lines. State both, and say which one solutions starting near $y = 1$ approach. [3]

Solutions

2.1 $y = 5$.

2.2 $y = 3$.

2.3 Away — at $y = 6$, $\frac{dy}{dx} = 1 > 0$, so y increases, moving away from 5.

3.1 **A** — an equilibrium has $\frac{dy}{dx} = 0$.

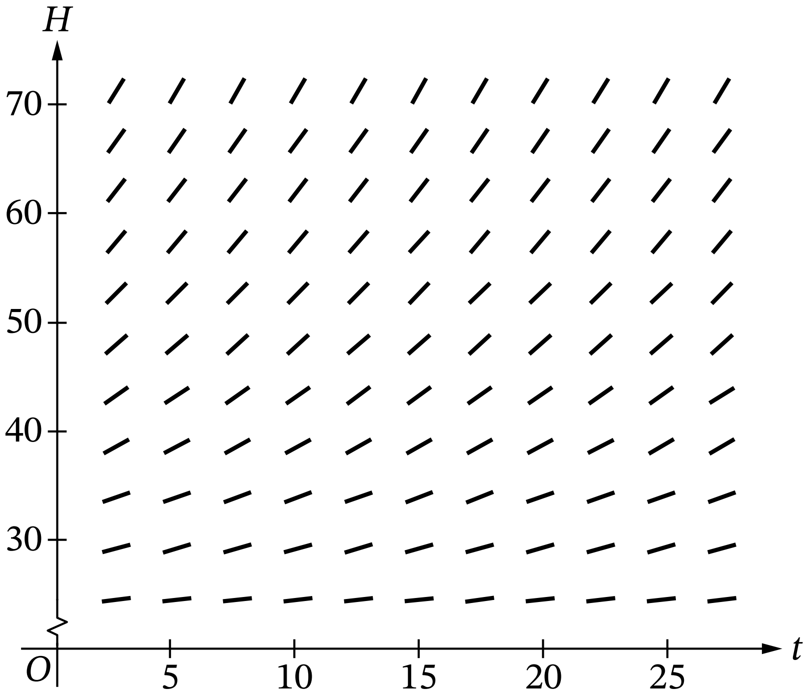
3.2 (a) $y = 2$. (b) Decreases — at $y = 5$, $\frac{dy}{dx} = 2 - 5 = -3 < 0$. (c) $y \to 2$.

3.3 Equilibria $y = 0$ and $y = 4$; a solution starting at $y = 1$ has $\frac{dy}{dx} = 1(3) > 0$, so it rises toward $y = 4$.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.



Part B

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Part C

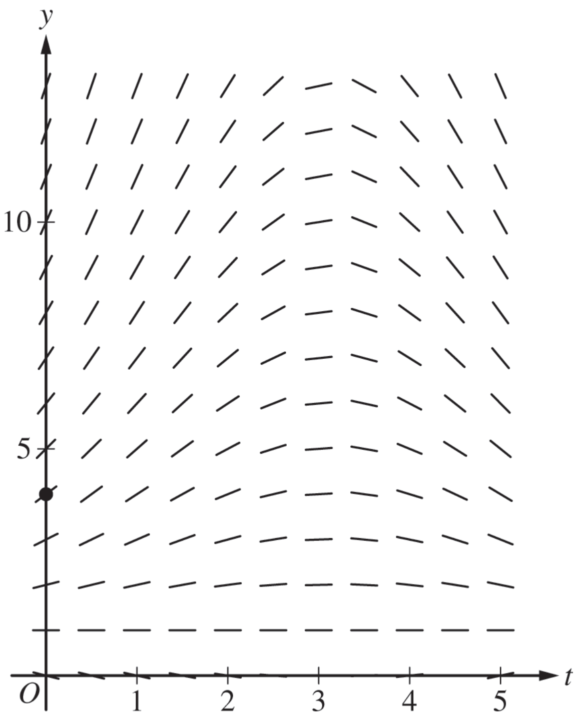
It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

7.5 Approximating Solutions Using Euler's Method

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

KEY WORDS euler's method 欧拉方法 • slope field 斜率场

Objective

Build the skills to answer exam questions on **Euler's method** — stepping along a slope field to approximate a solution.

You must be able to:

- apply the update $y_{\text{new}} = y_{\text{old}} + \dfrac{dy}{dx} \cdot \Delta x$
- carry out two or three steps in a table
- decide whether the estimate is an **over-** or **under-estimate** from concavity

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The Euler step

Starting from a known point, take a small step of size Δx and update using the local slope:

$$y_{\text{new}} = y_{\text{old}} + \left.\frac{dy}{dx}\right|_{\text{old}} \cdot \Delta x.$$

■ A worked two-step

Solve $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, with $\Delta x = 0.1$, to estimate $y(0.2)$.

- Step 1 at $(0, 1)$: slope $= 0 + 1 = 1$; $y \approx 1 + 1(0.1) = 1.1$ at $x = 0.1$.
- Step 2 at $(0.1, 1.1)$: slope $= 0.1 + 1.1 = 1.2$; $y \approx 1.1 + 1.2(0.1) = 1.22$ at $x = 0.2$.

So $y(0.2) \approx 1.22$.

- **A table helps**

Keep columns for x, y, slope $\frac{dy}{dx}$, and the next y. Smaller Δx gives a closer estimate.

- **Over- or under-estimate**

If the true solution is **concave up**, the tangent-line steps lie **below** it, so Euler's method **under-estimates**; concave down gives an over-estimate.

2 Practice

Now apply the methods above.

2.1 State the Euler update formula. [1]

2.2 For $\frac{dy}{dx} = 2x$, $y(1) = 3$, $\Delta x = 0.5$, find y at $x = 1.5$ (one step). [2]

2.3 Continue: from $(1.5, y)$ take one more step to estimate $y(2)$. [2]

3 Exam-style questions

3.1 Euler's method approximates a solution using [1]

- **A** the exact antiderivative
- **B** tangent-line steps along the slope field
- **C** a Riemann sum
- **D** the second derivative

3.2 Given $\dfrac{dy}{dx} = x - y$ with $y(0) = 2$ and $\Delta x = 0.5$, estimate $y(1)$ using two Euler steps. Show the table. [4]

3.3 For a solution that is concave up, state whether Euler's method over- or under-estimates the true value, and why. [2]

Solutions

2.1 $y_{\text{new}} = y_{\text{old}} + \dfrac{dy}{dx} \cdot \Delta x$.

2.2 slope at $(1, 3) = 2$; $y \approx 3 + 2(0.5) = 4$.

2.3 slope at $(1.5, 4) = 3$; $y \approx 4 + 3(0.5) = 5.5$.

3.1 **B** — it steps along tangent lines given by the slope field.

3.2 Step 1 at $(0, 2)$: slope $0 - 2 = -2$; $y \approx 2 + (-2)(0.5) = 1$ at $x = 0.5$. Step 2 at $(0.5, 1)$: slope $0.5 - 1 = -0.5$; $y \approx 1 + (-0.5)(0.5) = 0.75$ at $x = 1$.

3.3 Under-estimate — for a concave-up curve each tangent-line step lies below the true curve.

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

B. Write the second-degree Taylor polynomial for f about $x = 1$.

C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

AP Calculus BC: **2024**

x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

(a) For $x \ge 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

AP Calculus BC: **2016**

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

7.6 Finding General Solutions Using Separation of Variables

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS separable 可分离 • differential equation 微分方程

Objective

Build the skills to answer exam questions on **separation of variables** — the main method for solving a differential equation.

You must be able to:

- separate a **separable 可分离** equation so each variable is on its own side
- integrate both sides and include a single constant C
- solve for y where the question asks for an explicit solution

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Separating the variables

For $\dfrac{dy}{dx} = xy$: put all y with dy and all x with dx:

$$\frac{1}{y}\,dy = x\,dx.$$

■ Integrating both sides

$$\int \frac{1}{y}\,dy = \int x\,dx \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C.$$

One constant C is enough (combine the two).

■ Solving for y

Exponentiate: $|y| = e^{x^2/2+C} = e^C e^{x^2/2}$. Writing $A = \pm e^C$ (a new constant),

$$y = Ae^{x^2/2}.$$

■ **A quick separable example**

$\dfrac{dy}{dx} = \dfrac{x}{y}$: $y\,dy = x\,dx$, so $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$, i.e. $y^2 = x^2 + C_1$.

2 Practice

Now apply the methods above.

2.1 Separate the variables in $\dfrac{dy}{dx} = 2xy$. [1]

2.2 Solve $\dfrac{dy}{dx} = \dfrac{1}{y}$ (find the general solution). [3]

2.3 Solve $\dfrac{dy}{dx} = 3y$ (general solution in terms of a constant A). [3]

3 Exam-style questions

3.1 After separating $\dfrac{dy}{dx} = ky$, integrating the left side gives [1]

- **A** y
- **B** $\ln|y|$
- **C** $\dfrac{1}{y^2}$
- **D** e^y

3.2 Solve $\frac{dy}{dx} = xy^2$.

(a) Separate and integrate both sides. [3]

(b) Solve for y in terms of x and a constant. [2]

3.3 Solve $\frac{dy}{dx} = \frac{\cos x}{y}$, giving the general solution. [3]

Solutions

2.1 $\dfrac{1}{y}\,dy = 2x\,dx$.

2.2 $y\,dy = dx$; $\frac{1}{2}y^2 = x + C$; $y^2 = 2x + C_1$.

2.3 $\dfrac{1}{y}\,dy = 3\,dx$; $\ln|y| = 3x + C$; $y = Ae^{3x}$.

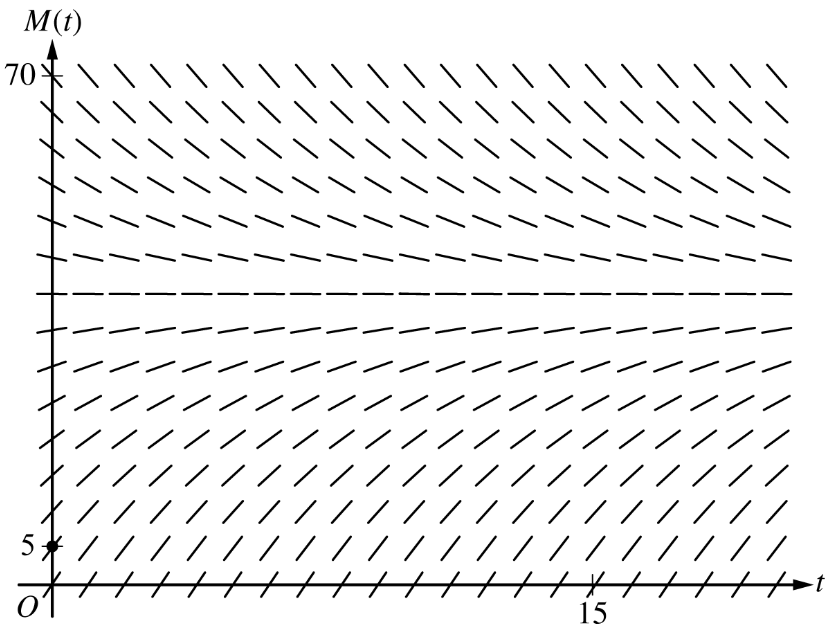
3.1 **B** — $\int \frac{1}{y}\,dy = \ln|y|$.

3.2 (a) $\dfrac{1}{y^2}\,dy = x\,dx$; $-\dfrac{1}{y} = \dfrac{x^2}{2} + C$. (b) $y = -\dfrac{1}{\frac{x^2}{2} + C} = \dfrac{-2}{x^2 + C_1}$.

3.3 $y\,dy = \cos x\,dx$; $\frac{1}{2}y^2 = \sin x + C$; $y^2 = 2\sin x + C_1$.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.

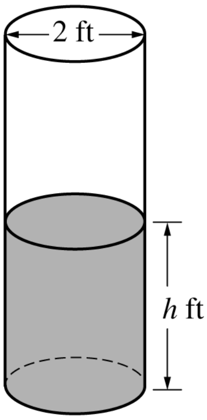


(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

7.7 Finding Particular Solutions Using Initial Conditions

Name: ______________ Class: __________ Date: ____________ **Total: 17 marks**

KEY WORDS initial condition 初始条件

Objective

Build the skills to answer exam questions on **particular solutions from initial conditions** — pinning down the constant C.

You must be able to:

- find the **general solution** by separation of variables
- substitute the **initial condition** 初始条件 to solve for C
- write the **particular solution** and use it to predict a value

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ General, then particular

Solve $\frac{dy}{dx} = 2x$ with $y(0) = 3$. Integrate: $y = x^2 + C$ (general). Apply the condition $y(0) = 3$: $3 = 0 + C$, so $C = 3$. Particular solution: $y = x^2 + 3$.

■ Finding C after an exponential solution

$\frac{dy}{dt} = ky$ has general solution $y = Ae^{kt}$. If $y(0) = 5$, then $5 = Ae^0 = A$, so $A = 5$ and $y = 5e^{kt}$. The initial value **is** the constant A here.

■ Substituting a non-zero start

Solve $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 2$. Separating gives $y^2 = x^2 + C_1$. At $(0, 2)$: $4 = 0 + C_1$, so $y^2 = x^2 + 4$, i.e. $y = \sqrt{x^2 + 4}$ (positive root, to match $y(0) = 2 > 0$).

■ Using the particular solution

Once you have $y = \sqrt{x^2 + 4}$, predict any value: $y(3) = \sqrt{9 + 4} = \sqrt{13}$.

2 Practice

Now apply the methods above.

2.1 Solve $\dfrac{dy}{dx} = 4x$ with $y(1) = 5$. [3]

2.2 For $y = Ae^{2t}$ with $y(0) = 7$, find A. [1]

2.3 Solve $\dfrac{dy}{dx} = 3x^2$ with $y(0) = -1$. [3]

3 Exam-style questions

3.1 The general solution $y = x^2 + C$ passes through $(2, 10)$. Then $C =$ [1]

- **A** 2
- **B** 6
- **C** 10
- **D** 14

3.2 A population satisfies $\dfrac{dP}{dt} = 0.5P$ with $P(0) = 200$.

(a) Write the general solution. [2]

(b) Find the particular solution using the initial condition. [1]

(c) Predict P at $t = 4$. [2]

3.3 Solve $\dfrac{dy}{dx} = \dfrac{6x}{y}$ with $y(0) = 4$, giving y explicitly. [4]

Solutions

2.1 $y = 2x^2 + C$; $5 = 2 + C \Rightarrow C = 3$; $y = 2x^2 + 3$.

2.2 $A = 7$.

2.3 $y = x^3 + C$; $-1 = 0 + C \Rightarrow C = -1$; $y = x^3 - 1$.

3.1 **B** — $10 = 4 + C \Rightarrow C = 6$.

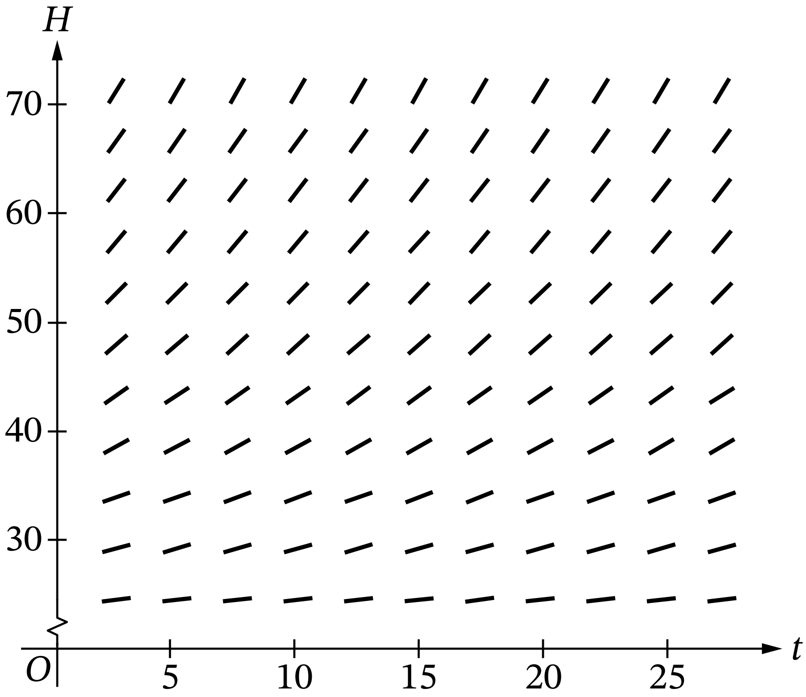
3.2 (a) $P = Ae^{0.5t}$. (b) $P(0) = A = 200$, so $P = 200e^{0.5t}$. (c) $P(4) = 200e^2 \approx 1478$.

3.3 $y\,dy = 6x\,dx$; $\frac{1}{2}y^2 = 3x^2 + C$; at $(0, 4)$: $8 = C$; $y = \sqrt{6x^2 + 16}$.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.



Part B

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Part C

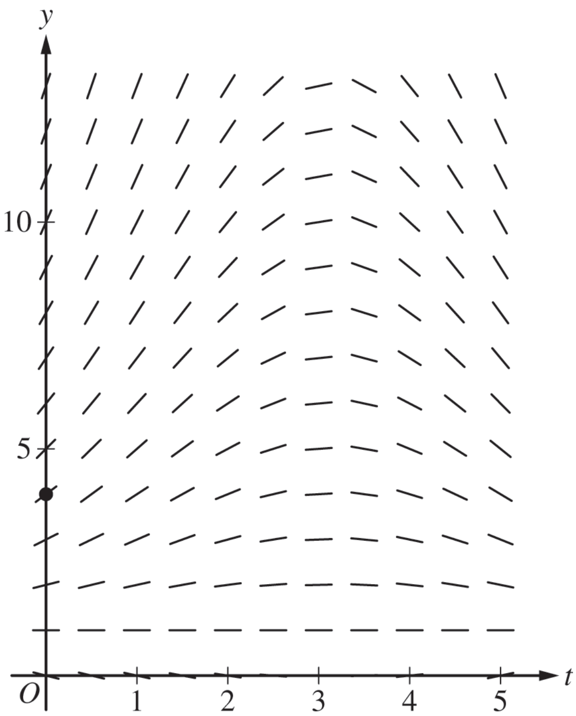
It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



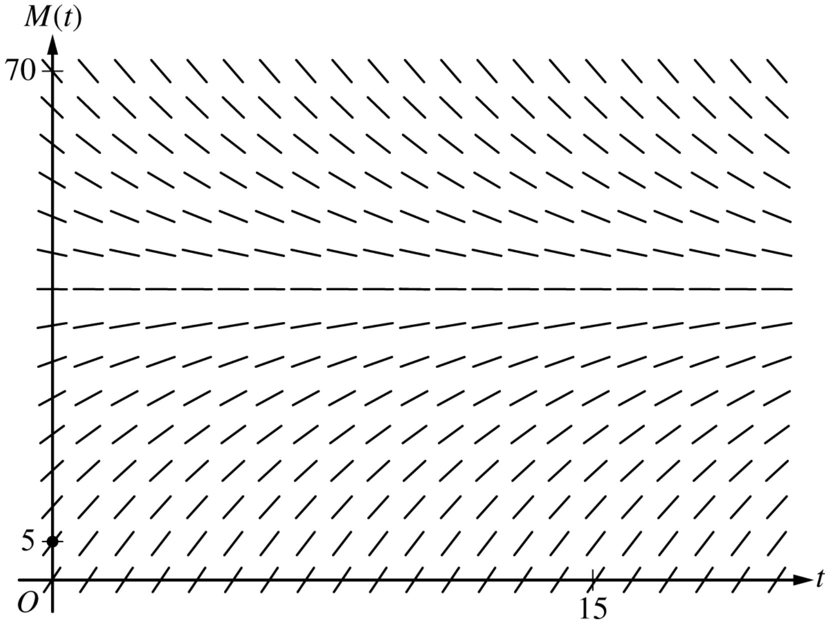
(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

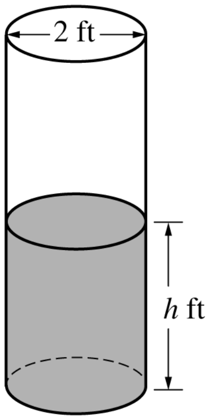
5. Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

   (a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

   (b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

   (c) Find the particular solution $y = f(x)$ to the differential equation $\dfrac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

   (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

   (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

   (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

7.8 Exponential Models with Differential Equations

Name: ____________ Class: ________ Date: __________ **Total: 17 marks**

Objective

Build the skills to answer exam questions on **exponential models** — growth and decay from $\frac{dy}{dt} = ky$.

You must be able to:

- write the solution $y = y_0e^{kt}$ from the model $\frac{dy}{dt} = ky$
- find k and y_0 from given data
- use the model to predict values and **half-life** / doubling situations

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The exponential solution

Any quantity whose rate of change is proportional to its size, $\frac{dy}{dt} = ky$, grows or decays exponentially:

$$y = y_0e^{kt},$$

where $y_0 = y(0)$ is the starting amount. $k > 0$ is growth; $k < 0$ is decay.

■ Finding y0 and k from data

A culture starts at 500 and is 2000 after 3 hours. Then $y_0 = 500$ and $2000 = 500e^{3k}$, so $e^{3k} = 4$, $k = \frac{1}{3}\ln 4 \approx 0.462$. Model: $y = 500e^{0.462t}$.

■ Decay and half-life

For decay, $k < 0$. A substance with $y = y_0e^{-0.1t}$ reaches **half** when $e^{-0.1t} = \frac{1}{2}$, so $-0.1t = \ln\frac{1}{2}$, $t = \frac{\ln 2}{0.1} \approx 6.93$ (the **half-life**).

■ Predicting a future value

With $y = 500e^{0.462t}$, the amount at $t = 5$ is $500e^{0.462(5)} = 500e^{2.31} \approx 5030$.

2 Practice

Now apply the methods above.

2.1 Write the solution of $\dfrac{dy}{dt} = 0.2y$ with $y(0) = 30$. [1]

2.2 A sample decays as $y = 80e^{-0.05t}$. Find the amount at $t = 10$. [2]

2.3 A population doubles: $y = y_0e^{kt}$ with $y_0 = 100$ and $y(2) = 200$. Find k. [3]

3 Exam-style questions

3.1 The solution of $\dfrac{dy}{dt} = ky$ with $y(0) = y_0$ is [1]

- **A** $y = y_0 + kt$
- **B** $y = y_0e^{kt}$
- **C** $y = kt^2$
- **D** $y = y_0k^t$

3.2 A radioactive sample follows $\dfrac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(a) Find k. [3]

(b) Predict the amount after 12 years. [2]

3.3 A bacteria count grows exponentially from 1000 to 8000 in 6 hours.

(a) Find the growth constant k. [2]

(b) How long, from the start, until the count reaches 16 000? [3]

Solutions

2.1 $y = 30e^{0.2t}$.

2.2 $80e^{-0.5} = 80(0.6065) \approx 48.5$.

2.3 $200 = 100e^{2k}$; $e^{2k} = 2$; $k = \frac{1}{2}\ln 2 \approx 0.347$.

3.1 B — $\dfrac{dy}{dt} = ky$ gives $y = y_0 e^{kt}$.

3.2 (a) $30 = 40e^{5k}$; $e^{5k} = 0.75$; $k = \frac{1}{5}\ln 0.75 \approx -0.0575$. (b) $A = 40e^{-0.0575(12)} = 40e^{-0.690} \approx 20.0$ mg.

3.3 (a) $8000 = 1000e^{6k} \Rightarrow e^{6k} = 8 \Rightarrow k = \frac{1}{6}\ln 8 \approx 0.347$. (b) $16000 = 1000e^{kt} \Rightarrow e^{kt} = 16 \Rightarrow t = \dfrac{\ln 16}{0.347} \approx 8$ h.

7.9 Logistic Models with Differential Equations

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS carrying capacity 环境容纳量 • logistic model 逻辑斯蒂模型

Objective

Build the skills to answer exam questions on the **logistic model** — growth limited by a carrying capacity.

You must be able to:

- read the **carrying capacity 环境容纳量** L from $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{L}\right)$
- state the long-run limit $\lim_{t\to\infty} P = L$
- find where growth is **fastest** $(P = \frac{L}{2})$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The logistic equation

Real growth slows as resources run out. The logistic model is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right),$$

where L is the **carrying capacity** — the population level the system can sustain.

Reading L off the equation

L is the value that makes the bracket zero. From $\dfrac{dP}{dt} = 0.03P\left(1 - \dfrac{P}{800}\right)$, the carrying capacity is $L = 800$.

The long-run limit

Whatever the (positive) start, P approaches L: $\lim_{t\to\infty} P = L$. Here $P \to 800$. If P starts above L it decreases toward L; below L it increases toward L.

■ **Fastest growth at L/2**

The growth rate $\frac{dP}{dt}$ is largest at the **inflection point** $P = \frac{L}{2}$ — the middle of the S-shaped curve. Here that is $P = 400$.

2 Practice

Now apply the methods above.

2.1 For $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{2000}\right)$, state the carrying capacity. [1]

2.2 For the same model, state $\lim_{t\to\infty} P$ (given $P(0) = 100$). [1]

2.3 At what population is growth fastest for that model? [1]

3 Exam-style questions

3.1 In $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$, growth is fastest when $P =$ [1]

- **A** 0
- **B** $\frac{L}{4}$
- **C** $\frac{L}{2}$
- **D** L

3.2 A fish population follows $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right)$, with $P(0) = 500$.

(a) State the carrying capacity. [1]

(b) State $\lim\limits_{t\to\infty} P$. [1]

(c) Find the population at which the population grows fastest. [1]

3.3 A logistic model has carrying capacity 1200 and $P(0) = 1500$. State, with a reason, whether the population increases or decreases toward the carrying capacity. [2]

Solutions

2.1 $L = 2000$.

2.2 2000.

2.3 $P = 1000$ (i.e. $L/2$).

3.1 **C** — fastest growth at the inflection point $P = \frac{L}{2}$.

3.2 (a) 5000. (b) 5000. (c) 2500.

3.3 Decreases — $P(0) = 1500 > 1200 = L$, so $1 - \frac{P}{L} < 0$ and $\dfrac{dP}{dt} < 0$, pulling P down toward 1200.