Week 15

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

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AP Calculus BC Name: ____________ Score: ______

1. A tap runs at 3 L/min for 4 min. The area under the rate graph (accumulated volume) is...

   ____

2. For $f(x) = x^2$ on $[0, 2]$ with 2 right rectangles (width 1): heights $f(1) = 1$, $f(2) = 4$. Find the right Riemann sum.

   ____

3. For n equal strips on $[a, b]$, the width Δx is...
   - ☐ $\dfrac{b-a}{n}$
   - ☐ $b - a$
   - ☐ $\dfrac{n}{b-a}$
   - ☐ $n(b - a)$

4. By FTC Part 1, $\dfrac{d}{dx} \int_a^x f(t)\,dt =$
   - ☐ $F(x) - F(a)$
   - ☐ $f(x)$
   - ☐ $f'(x)$
   - ☐ $f(a)$

5. For $g(x) = \int_a^x f(t)\,dt$, g is increasing exactly where...
   - ☐ $f(x) > 0$
   - ☐ $f(x) < 0$
   - ☐ $f'(x) > 0$
   - ☐ $g(x) > 0$

6. Given $\int_0^4 f\,dx = 10$ and $\int_0^4 g\,dx = 3$, find $\int_0^4 (2f - g)\,dx$.

   ____

7. FTC Part 2 says $\int_a^b f\,dx =$
   - ☐ $f'(b) - f'(a)$
   - ☐ $F(b) - F(a)$ where $F' = f$
   - ☐ $F(a) - F(b)$
   - ☐ $f(b) - f(a)$

8. What is $\displaystyle\int x^3\,dx$?

- ☐ $3x^2 + C$
- ☐ $\dfrac{x^4}{4} + C$
- ☐ $x^4 + C$
- ☐ $\dfrac{x^2}{2} + C$

9. An integral is improper when...

- ☐ a limit of integration is infinite
- ☐ the integrand is unbounded on the interval
- ☐ the integrand is a polynomial
- ☐ the interval is finite and the function bounded

Tick every correct option.

10. Where a rate graph dips below the axis, that area subtracts from the net accumulated change.

☐ True ☐ False

AP Calculus BC

1. **12**

   Rectangle area $= 3 \times 4 = 12$ L.

2. **5**

   $1 \cdot 1 + 4 \cdot 1 = 5$.

3. $\dfrac{b-a}{n}$

   The interval width divided by the number of strips.

4. $f(x)$

   The derivative of the accumulation function is the integrand $f(x)$.

5. $f(x) > 0$

   $g' = f$, so g increases where $f > 0$.

6. **17**

   $2(10) - 3 = 17$.

7. $F(b) - F(a)$ **where** $F' = f$

   Evaluate an antiderivative at upper minus lower limit.

8. $\dfrac{x^4}{4} + C$

   Add one to the exponent, divide by it: $\frac{x^4}{4} + C$.

9. **a limit of integration is infinite; the integrand is unbounded on the interval**

   Infinite limit or unbounded integrand $\to$ improper.

10. **True**

    Below-axis area is negative (signed area).

6.1 Exploring Accumulations of Change

Name: ____________ Class: ________ Date: __________ **Total: 10 marks**

KEY WORDS accumulated change 累积变化 • accumulation 累积

Objective

Build the skills to answer exam questions on **accumulation of change** — reading a total change as the area under a rate graph.

You must be able to:

- interpret the **area** under a rate-of-change graph as an **accumulated change** 累积变化
- treat area **above** the axis as positive and area **below** as negative
- attach correct **units** (rate unit × time unit) to an accumulation

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Area under a rate graph is a total

If water flows in at a **rate** $R(t)$ litres per minute, the total water added between $t = 0$ and $t = 5$ is the **area** under the R-vs-t graph over $[0, 5]$. A rate graph that is a rectangle of height 3 L/min over 5 min accumulates $3 \times 5 = 15$ L.

■ Splitting into simple shapes

If the rate graph is made of straight lines, find the area with triangles and rectangles. A rate rising linearly from 0 to 4 L/min over 6 min is a triangle: area $= \frac{1}{2}(6)(4) = 12$ L.

■ Signed area

When a rate goes **negative** (water leaving), that area counts as a **loss**. Over a stretch where the graph is $+8$ area then -3 area, the net accumulated change is $8 - 3 = +5$.

■ Units

The accumulation's unit is (rate unit) × (time unit). A rate in L/min over minutes accumulates **litres**; a velocity in m/s over seconds accumulates **metres**.

2 Practice

Now apply the methods above.

2.1 A tap runs at a constant 2.5 L/min for 8 minutes. How much water flows out? [1]

2.2 A rate graph is a triangle: the rate rises linearly from 0 to 10 units/s over 4 s. Find the accumulated change. [2]

2.3 Over one interval a quantity gains an area of 12 and over the next it loses an area of 5. State the net accumulated change. [1]

3 Exam-style questions

3.1 The area under a velocity-vs-time graph represents [1]

- **A** acceleration
- **B** displacement
- **C** speed
- **D** force

3.2 A machine consumes power at a rate $P(t)$ kilowatts, shown as a graph made of straight segments: a rectangle of height 2 over $0 \leq t \leq 3$ hours, then a triangle rising from 0 to 4 over $3 \leq t \leq 5$ hours.

(a) Find the energy used over $0 \leq t \leq 3$. [1]

(b) Find the energy used over $3 \leq t \leq 5$. [2]

(c) State the total energy used, with units. [1]

3.3 Sand is added to a pile at a rate that gives an accumulated area of 30 (kg) over the first hour, then removed at a rate giving an area of 8 (kg) over the next hour. State the net change in the pile's mass. [1]

Solutions

2.1 $2.5 \times 8 = 20$ L.

2.2 Triangle area $= \frac{1}{2}(4)(10) = 20$ units.

2.3 $12 - 5 = 7$.

3.1 **B** — the area under a velocity–time graph is displacement.

3.2 (a) $2 \times 3 = 6$ kWh. (b) triangle $\frac{1}{2}(2)(4) = 4$ kWh. (c) $6 + 4 = 10$ kWh.

3.3 $30 - 8 = 22$ kg.

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\,dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

6.2 Approximating Areas with Riemann Sums

Name: ______________ Class: __________ Date: ____________ **Total: 12 marks**

KEY WORDS riemann sum 黎曼和 • trapezoidal 梯形 • trapezoidal sum 梯形法 • midpoint 中点 • integral 积分

Objective

Build the skills to answer exam questions on **Riemann sums** — estimating the area under a curve with rectangles.

You must be able to:

- compute **left**, **right**, and **midpoint** 中点 Riemann sums from a function or a table
- compute a **trapezoidal** 梯形 estimate
- decide whether an estimate **over-** or **under-estimates** using increasing/decreasing or concavity

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Left and right sums

Estimate $\int_0^6 f\,dx$ with 3 equal subintervals of width $\Delta x = 2$, using the table:

x	**0**	**2**	**4**	**6**
$f(x)$	1	4	9	16

- **Left** sum uses the left endpoints: $2(1+4+9) = 28$.
- **Right** sum uses the right endpoints: $2(4+9+16) = 58$.

Midpoint sum

The **midpoint** sum uses the value at the middle of each subinterval. With $f(x) = x^2$ on $[0, 6]$, $n = 3$, $\Delta x = 2$, midpoints $1, 3, 5$: $2\big(f(1)+f(3)+f(5)\big) = 2(1+9+25) = 70$.

■ **Trapezoidal estimate**

A **trapezoidal** sum averages each pair of endpoint heights: $\Delta x\left[\frac{f_0}{2}+f_1+f_2+\frac{f_3}{2}\right]$. For the table: $2\left[\frac{1}{2}+4+9+\frac{16}{2}\right]=2(21.5)=43$. It equals the average of the left and right sums.

■ **Over- or under-estimate**

For an **increasing** function, a left sum **under**-estimates and a right sum **over**-estimates. For a **concave-up** function, the trapezoidal sum **over**-estimates and the midpoint sum **under**-estimates.

2 Practice

Now apply the methods above.

Use the table for 2.1–2.3:

x	**0**	**3**	**6**	**9**
$g(x)$	2	5	6	10

2.1 Find the **left** Riemann sum for $\displaystyle\int_0^9 g\,dx$ with the three subintervals shown. [2]

2.2 Find the **right** Riemann sum for the same integral. [2]

2.3 Find the **trapezoidal** estimate. [2]

3 Exam-style questions

3.1 For an increasing function, a **left** Riemann sum gives an [1]

- **A** over-estimate
- **B** under-estimate
- **C** exact value

- **D** value that depends on concavity

3.2 A car's speed $v(t)$ in m/s is recorded:

t **(s)**	**0**	**4**	**8**	**12**
$v(t)$	0	12	20	24

(a) Estimate the distance travelled over $0 \le t \le 12$ using a **right** Riemann sum with three subintervals. [2]

(b) Is this an over- or under-estimate of the true distance? Give a reason. [1]

3.3 The function f is concave up on $[0, 4]$. A student estimates $\int_0^4 f\,dx$ with a trapezoidal sum. State, with a reason, whether the estimate is too large or too small. [2]

Solutions

2.1 $\Delta x = 3$; left $= 3(2+5+6) = 39$.

2.2 right $= 3(5+6+10) = 63$.

2.3 $3\left[\frac{2}{2}+5+6+\frac{10}{2}\right] = 3(17) = 51$. (Equivalently the average of 39 and 63.)

3.1 B — for an increasing function the left sum under-estimates.

3.2 (a) $4(12+20+24) = 224$ m. (b) Over-estimate — v is increasing, so right endpoints are the largest values on each subinterval.

3.3 Too large — for a concave-up curve each trapezoid lies **above** the curve, so the trapezoidal sum over-estimates.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

(d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

   (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

   (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

   (d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1 + x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\,dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\,dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

   (a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

   (b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

   (c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

   (d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

6.3 Riemann Sums, Summation Notation, and Definite Integral Notation

Name: ______________ Class: __________ Date: ____________ **Total: 11 marks**

KEY WORDS definite integral 定积分 • riemann sum 黎曼和 • integral 积分 • antiderivative 原函数

Objective

Build the skills to answer exam questions on **Riemann sums and definite-integral notation** — connecting a limit of sums to $\int_a^b f\,dx$.

You must be able to:

- write a Riemann sum with **summation notation** $\sum$
- recognise the **definite integral** 定积分 as the limit of Riemann sums as $n \to \infty$
- read the meaning of the parts of $\displaystyle\int_a^b f(x)\,dx$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Summation notation for a sum

A right Riemann sum with n equal pieces on $[a, b]$ has width $\Delta x = \dfrac{b-a}{n}$ and sample points $x_k = a + k\,\Delta x$:

$$\sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The definite integral as a limit

Letting the rectangles become infinitely thin gives the exact area:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

The integral **is** this limit — a sum of infinitely many infinitely thin pieces.

■ **Reading the notation**

In $\int_a^b f(x)\,dx$: a and b are the **limits of integration**, $f(x)$ is the **integrand**, and dx marks the variable and the "width" of each piece. The result is a **number** (a signed area), not a function.

■ **From a limit back to an integral**

A limit like $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{1+\frac{3k}{n}}\cdot\frac{3}{n}$ is a right sum with $\Delta x=\frac{3}{n}$ on $[1,4]$, so it equals $\int_1^4 \sqrt{x}\,dx$.

2 Practice

Now apply the methods above.

2.1 For $\int_2^{10} f\,dx$ estimated with $n=4$ equal subintervals, state Δx. [1]

2.2 Write, in summation notation, a right Riemann sum for $\int_0^2 f\,dx$ with n equal subintervals. [2]

2.3 In $\int_1^5 (3x+2)\,dx$, name the integrand and the limits of integration. [2]

3 Exam-style questions

3.1 The definite integral $\int_a^b f\,dx$ is defined as [1]

- **A** an antiderivative of f
- **B** the limit of Riemann sums as $n\to\infty$
- **C** the slope of f
- **D** the average of $f(a)$ and $f(b)$

3.2 Consider $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{4k}{n}\right)^2 \cdot \frac{4}{n}$.

(a) State the width Δx and the interval $[a, b]$. [2]

(b) Write the limit as a definite integral. [2]

3.3 A definite integral $\displaystyle\int_0^6 v(t)\,dt$ evaluates to a negative number. Explain what a **negative** value tells you about the signed area, in one sentence. [1]

Solutions

2.1 $\Delta x = \dfrac{10-2}{4} = 2.$

2.2 $\Delta x = \dfrac{2}{n}$, $x_k = \dfrac{2k}{n}$; sum $= \sum_{k=1}^{n} f\left(\frac{2k}{n}\right) \cdot \frac{2}{n}$.

2.3 Integrand $3x+2$; limits of integration 1 and 5.

3.1 **B** — the definite integral is the limit of Riemann sums.

3.2 (a) $\Delta x = \dfrac{4}{n}$, so $b-a=4$; the sample $2+\frac{4k}{n}$ starts at 2, so $[a,b]=[2,6]$. (b) $\displaystyle\int_2^6 x^2\,dx$.

3.3 More signed area lies **below** the axis than above it over $[0,6]$.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0, 10]$, $[10, 20]$, and $[20, 30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t + 15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

6.4 The Fundamental Theorem of Calculus and Accumulation Functions

Name: ____________ Class: ________ Date: __________ **Total: 11 marks**

KEY WORDS accumulation function 累积函数 • accumulation 累积 • integral 积分

Objective

Build the skills to answer exam questions on the **Fundamental Theorem of Calculus and accumulation functions** — differentiating an integral with a variable limit.

You must be able to:

- use $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ (the **accumulation function** 累积函数)
- apply the **chain rule** when the upper limit is a function of x
- evaluate an accumulation function $g(x) = \displaystyle\int_a^x f(t)\,dt$ at given points

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The Fundamental Theorem (derivative form)

If $g(x) = \displaystyle\int_a^x f(t)\,dt$, then $g'(x) = f(x)$. Differentiating "undoes" the integral. For $g(x) = \displaystyle\int_2^x \cos(t^2)\,dt$, we get $g'(x) = \cos(x^2)$ — no integration needed.

Chain rule with a variable upper limit

If the upper limit is $u(x)$, then

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f\big(u(x)\big)\cdot u'(x).$$

For $\displaystyle\int_0^{x^2} \sin t\,dt$: the answer is $\sin(x^2)\cdot 2x$.

■ **Evaluating an accumulation function**

$g(x) = \int_1^x (2t)\,dt$. A value like $g(3) = \int_1^3 2t\,dt = [t^2]_1^3 = 9 - 1 = 8$, and $g(1) = 0$ (zero-width interval).

■ **Sign of the accumulation**

g **increases** where $f > 0$ and **decreases** where $f < 0$, because $g' = f$. Reading a graph of f tells you where g rises and falls.

2 Practice

Now apply the methods above.

2.1 If $g(x) = \int_0^x (t^3 + 1)\,dt$, find $g'(x)$. [1]

2.2 Find $\frac{d}{dx}\int_1^x \sqrt{t}\,dt$. [1]

2.3 Find $\frac{d}{dx}\int_0^{x^3} \cos t\,dt$. [2]

3 Exam-style questions

3.1 If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) =$ [1]

- **A** $f'(x)$
- **B** $f(x)$
- **C** $f(x) - f(a)$
- **D** $F(x) - F(a)$

3.2 Let $g(x) = \int_2^x f(t)\,dt$, where f is continuous.

(a) Find $g(2)$. [1]

(b) Given $f(5) = 7$, find $g'(5)$. [1]

(c) If $f(t) > 0$ for all $t > 2$, state whether g is increasing or decreasing for $x > 2$, with a reason. [2]

3.3 Let $h(x) = \displaystyle\int_0^{2x} e^{-t^2}\, dt$. Find $h'(x)$. [2]

Solutions

2.1 $g'(x) = x^3 + 1$.

2.2 $\sqrt{x}$.

2.3 $\cos(x^3) \cdot 3x^2$.

3.1 **B** — by the Fundamental Theorem, $g'(x) = f(x)$.

3.2 (a) $g(2) = 0$ (zero-width interval). (b) $g'(5) = f(5) = 7$. (c) Increasing — $g'(x) = f(x) > 0$ for $x > 2$.

3.3 $h'(x) = e^{-(2x)^2} \cdot 2 = 2e^{-4x^2}$.

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

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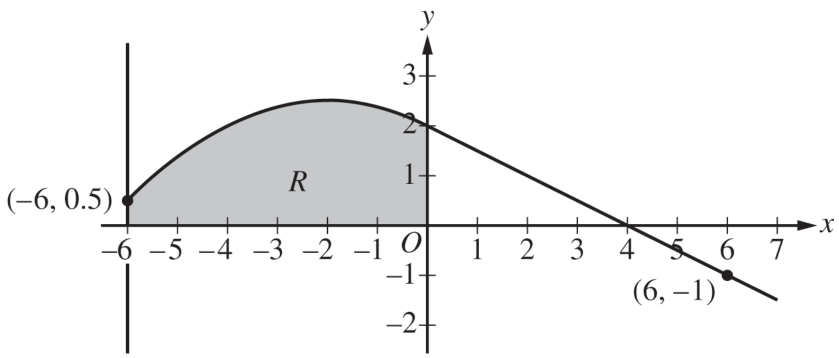


Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each in the space provided for that part.

x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

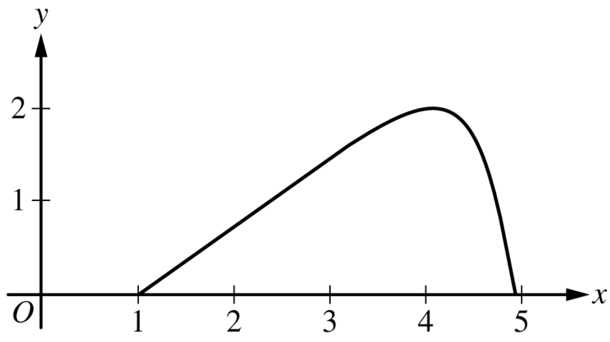
   (a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

   (b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

   (c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

   (d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



2. For $0 \le t \le \pi$, a particle is moving along the curve shown so that its position at time t is $(x(t), y(t))$, where $x(t)$ is not explicitly given and $y(t) = 2\sin t$. It is known that $\dfrac{dx}{dt} = e^{\cos t}$. At time $t = 0$, the particle is at position $(1, 0)$.

(a) Find the acceleration vector of the particle at time $t = 1$. Show the setup for your calculations.

(b) For $0 \le t \le \pi$, find the first time t at which the speed of the particle is 1.5. Show the work that leads to your answer.

(c) Find the slope of the line tangent to the path of the particle at time $t = 1$. Find the x-coordinate of the position of the particle at time $t = 1$. Show the work that leads to your answers.

(d) Find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. Show the setup for your calculations.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

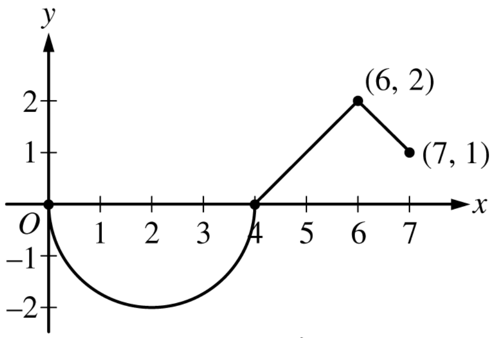
   (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

   (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

   (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\,dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

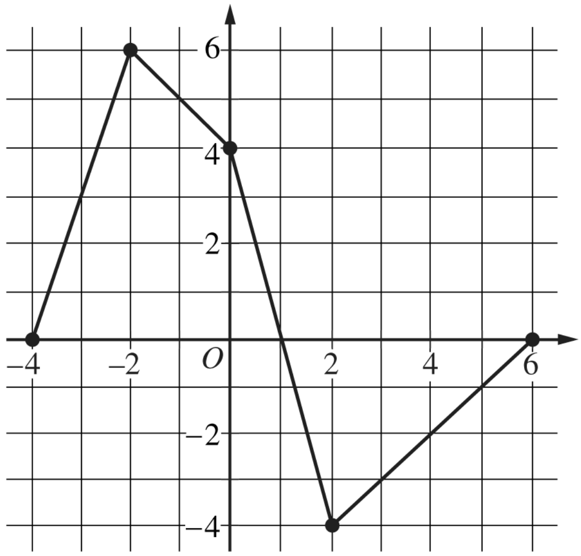
(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

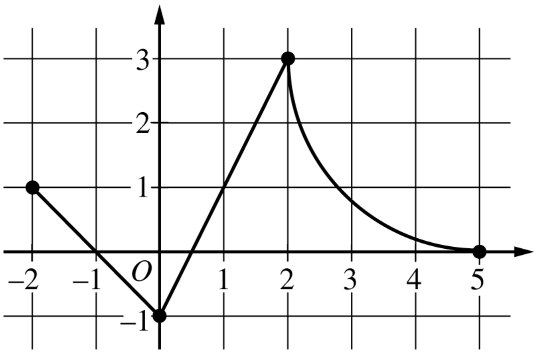
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

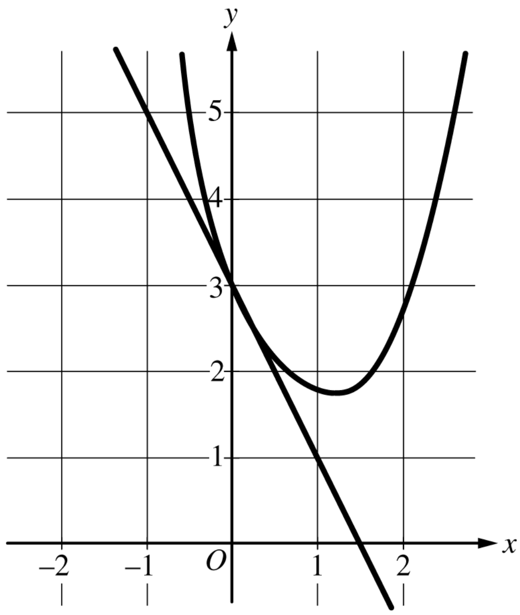
3. The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\dfrac{t}{100}\right)^3\left(1 - \dfrac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

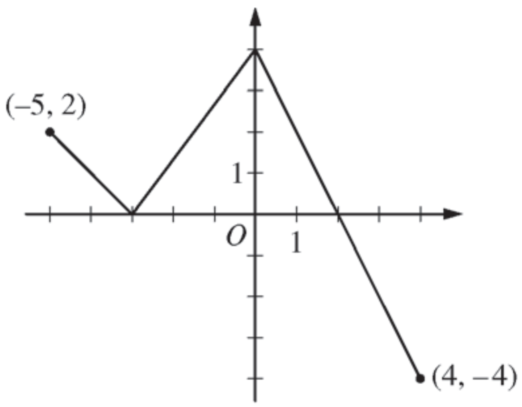
CALCULUS BC

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

6.5 Interpreting the Behavior of Accumulation Functions

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS local extrema 极值 • accumulation function 累积函数 • accumulation 累积

Objective

Build the skills to answer exam questions on the **behavior of accumulation functions** — reading features of $g(x) = \int_a^x f(t)\,dt$ from a graph of f.

You must be able to:

- decide where g is **increasing/decreasing** from the sign of f
- locate **local extrema 极值** of g where f crosses zero (sign change)
- decide **concavity** of g from whether f is increasing or decreasing

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ g rises where f is positive

Because $g'(x) = f(x)$, the accumulation function g **increases** on intervals where the graph of f is **above** the axis and **decreases** where f is **below** the axis.

■ Extrema of g at zeros of f

g has a **local maximum** where f changes from $+$ to $-$, and a **local minimum** where f changes from $-$ to $+$. A zero of f where the sign does **not** change is not an extremum of g.

■ Concavity of g from the slope of f

$g''(x) = f'(x)$, so g is **concave up** where f is **increasing** and **concave down** where f is **decreasing**. A point where f has a local max or min is a **point of inflection** of g.

■ Comparing values of g

To compare $g(3)$ and $g(0)$, look at the net signed area of f between them: $g(3)-g(0) = \int_0^3 f\,dt$. More area above the axis than below makes $g(3) > g(0)$.

2 Practice

Now apply the methods above. Let $g(x) = \displaystyle\int_0^x f(t)\,dt$, where f is continuous.

2.1 On an interval where $f(t) < 0$, state whether g is increasing or decreasing. [1]

2.2 At a point where f changes from positive to negative, what feature does g have? [1]

2.3 On an interval where f is increasing, state the concavity of g. [1]

3 Exam-style questions

3.1 $g(x) = \int_0^x f(t)\,dt$. The graph of f crosses from below the axis to above it at $x = 4$. At $x = 4$, g has a [1]

- **A** local maximum
- **B** local minimum
- **C** point of inflection
- **D** vertical asymptote

3.2 The graph of f consists of straight segments: $f > 0$ on $(0, 3)$, $f < 0$ on $(3, 7)$, and $f(3) = 0$. Let $g(x) = \int_0^x f(t)\,dt$.

(a) On what interval is g increasing? [1]

(b) At what x does g attain a local maximum? Justify. [2]

3.3 For $g(x) = \int_0^x f(t)\,dt$, the graph of f is increasing on $(1, 5)$. State, with a reason, the concavity of g on $(1, 5)$. [2]

Solutions

2.1 Decreasing — $g'(x) = f(x) < 0$.

2.2 A local maximum.

2.3 Concave up — $g''(x) = f'(x) > 0$.

3.1 B — f changes $-$ to $+$, so g' changes sign the same way and g has a local minimum.

3.2 (a) g increases on $(0, 3)$ where $f > 0$. (b) At $x = 3$, because f (which is g') changes from positive to negative there.

3.3 Concave up — $g''(x) = f'(x) > 0$ since f is increasing.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0, 10]$, $[10, 20]$, and $[20, 30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16 \sin\left(\frac{\pi}{20}(t + 15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

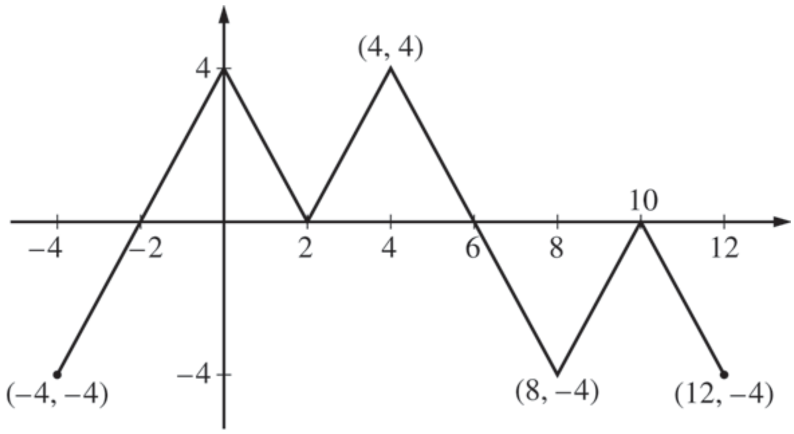
Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

CALCULUS BC
SECTION II, Part B
Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

6.6 Applying Properties of Definite Integrals

Name: ______ Class: ______ Date: ______ **Total: 10 marks**

KEY WORDS integral 积分 • definite integral 定积分

Objective

Build the skills to answer exam questions on the **properties of definite integrals** — combining and manipulating integral values.

You must be able to:

- use $\int_a^a f = 0$ and $\int_b^a f = -\int_a^b f$
- **split** an integral over adjacent intervals: $\int_a^c f = \int_a^b f + \int_b^c f$
- factor out constants and add integrands: $\int_a^b (cf + g) = c\int_a^b f + \int_a^b g$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Zero-width and reversing

$\int_a^a f\,dx = 0$ (no width). Reversing the limits flips the sign: $\int_b^a f\,dx = -\int_a^b f\,dx$.

■ Adding adjacent intervals

Integrals join end-to-end: $\int_1^4 f = \int_1^3 f + \int_3^4 f$. If $\int_1^3 f = 5$ and $\int_1^4 f = 9$, then $\int_3^4 f = 9 - 5 = 4$.

■ Linearity

Constants come out and sums split:

$$\int_a^b \big(3f(x) + 2g(x)\big)\,dx = 3\int_a^b f + 2\int_a^b g.$$

If $\int_0^2 f = 4$ and $\int_0^2 g = 1$, then $\int_0^2 (3f + 2g) = 3(4) + 2(1) = 14$.

■ **Combining with a reversal**

Given $\int_2^5 f = 10$, find $\int_5^2 3f$. Reverse and factor: $\int_5^2 3f = 3\int_5^2 f = 3(-10) = -30$.

2 Practice

Now apply the methods above. You are given $\displaystyle\int_0^3 f = 6$, $\displaystyle\int_3^7 f = -2$, and $\displaystyle\int_0^3 g = 5$.

2.1 Find $\displaystyle\int_0^7 f\,dx$. [1]

2.2 Find $\displaystyle\int_3^0 f\,dx$. [1]

2.3 Find $\displaystyle\int_0^3 (2f + g)\,dx$. [2]

3 Exam-style questions

3.1 $\displaystyle\int_5^5 f(x)\,dx =$ [1]

- **A** $f(5)$
- **B** 1
- **C** 0
- **D** undefined

3.2 You are given $\displaystyle\int_1^6 h = 12$ and $\displaystyle\int_1^4 h = 7$.

(a) Find $\displaystyle\int_4^6 h\,dx$. [2]

(b) Find $\displaystyle\int_6^1 h\,dx$. [1]

3.3 Given $\displaystyle\int_0^2 p = 8$ and $\displaystyle\int_0^2 q = -3$, find $\displaystyle\int_0^2 (5p - 4q)\,dx$. [2]

Solutions

2.1 $\int_0^7 f = \int_0^3 f + \int_3^7 f = 6 + (-2) = 4.$

2.2 $\int_3^0 f = -\int_0^3 f = -6.$

2.3 $2(6) + 5 = 17.$

3.1 **C** — a zero-width interval gives 0.

3.2 (a) $\int_4^6 h = \int_1^6 h - \int_1^4 h = 12 - 7 = 5.$ (b) $\int_6^1 h = -12.$

3.3 $5(8) - 4(-3) = 40 + 12 = 52.$

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

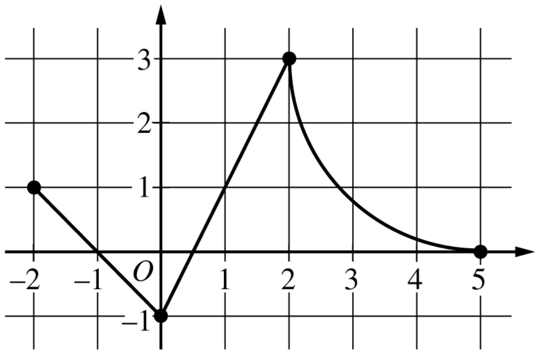
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

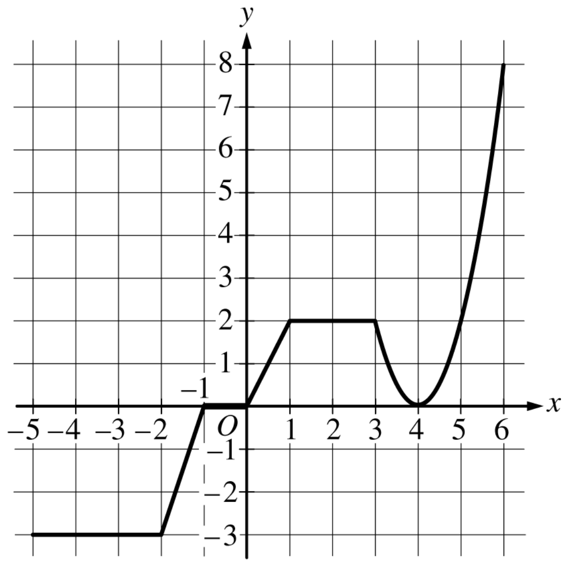
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

   (a) If $f(1) = 3$, what is the value of $f(-5)$?

   (b) Evaluate $\int_1^6 g(x)\,dx$.

   (c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

   (d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

6.7 The Fundamental Theorem of Calculus and Definite Integrals

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS net change 净变化 • antiderivative 原函数 • integral 积分 • definite integral 定积分

Objective

Build the skills to answer exam questions on the **Fundamental Theorem of Calculus (evaluation form)** — using an antiderivative to evaluate a definite integral.

You must be able to:

- apply $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F' = f$
- evaluate integrals of powers, exponentials, and basic trig functions
- interpret $\int_a^b f'(x)\,dx = f(b) - f(a)$ as a **net change** 净变化

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Evaluating with an antiderivative

$\int_1^3 2x\,dx$: an antiderivative of $2x$ is x^2, so

$$\int_1^3 2x\,dx = \left[x^2\right]_1^3 = 3^2 - 1^2 = 8.$$

A power and a constant

$\int_0^2 (3x^2+1)\,dx = \left[x^3+x\right]_0^2 = (8+2) - 0 = 10$. Antidifferentiate term by term, then subtract the values at the limits.

Net change from a rate

If f' is a rate of change, then $\int_a^b f'(x)\,dx = f(b) - f(a)$ is the **net change** in f. If a tank's level changes at rate $r(t)$ and $\int_0^{10} r\,dt = -4$, the level **fell** by 4 over those 10 units.

■ **Trig and exponential**

$\int_0^{\pi} \sin x\,dx = \big[-\cos x\big]_0^{\pi} = -\cos\pi + \cos 0 = 1+1 = 2$, and $\int_0^1 e^x\,dx = \big[e^x\big]_0^1 = e-1$.

2 Practice

Now apply the methods above.

2.1 Evaluate $\displaystyle\int_0^4 x\,dx$. [2]

2.2 Evaluate $\displaystyle\int_1^2 (4x^3)\,dx$. [2]

2.3 Evaluate $\displaystyle\int_0^{\pi/2} \cos x\,dx$. [2]

3 Exam-style questions

3.1 $\displaystyle\int_a^b f'(x)\,dx =$ [1]

- **A** $f'(b) - f'(a)$
- **B** $f(b) - f(a)$
- **C** $f(b) + f(a)$
- **D** $\frac{1}{2}(f(a) + f(b))$

3.2 A particle's velocity is $v(t) = 3t^2 - 4$ m/s.

(a) Evaluate $\displaystyle\int_0^2 v(t)\,dt$. [2]

(b) State what this value represents physically. [1]

3.3 Water enters a tank at rate $r(t) = 6t$ L/min. Find the total volume added between $t = 1$ and $t = 4$ minutes. [3]

Solutions

2.1 $\left[\frac{1}{2}x^2\right]_0^4 = 8 - 0 = 8.$

2.2 $\left[x^4\right]_1^2 = 16 - 1 = 15.$

2.3 $\left[\sin x\right]_0^{\pi/2} = 1 - 0 = 1.$

3.1 **B** — by the FTC, $\int_a^b f' = f(b) - f(a)$.

3.2 (a) $\left[t^3 - 4t\right]_0^2 = (8 - 8) - 0 = 0$. (b) The particle's **displacement** over $0 \le t \le 2$ is 0 — it returns to its start.

3.3 $\displaystyle\int_1^4 6t\,dt = \left[3t^2\right]_1^4 = 48 - 3 = 45$ L.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

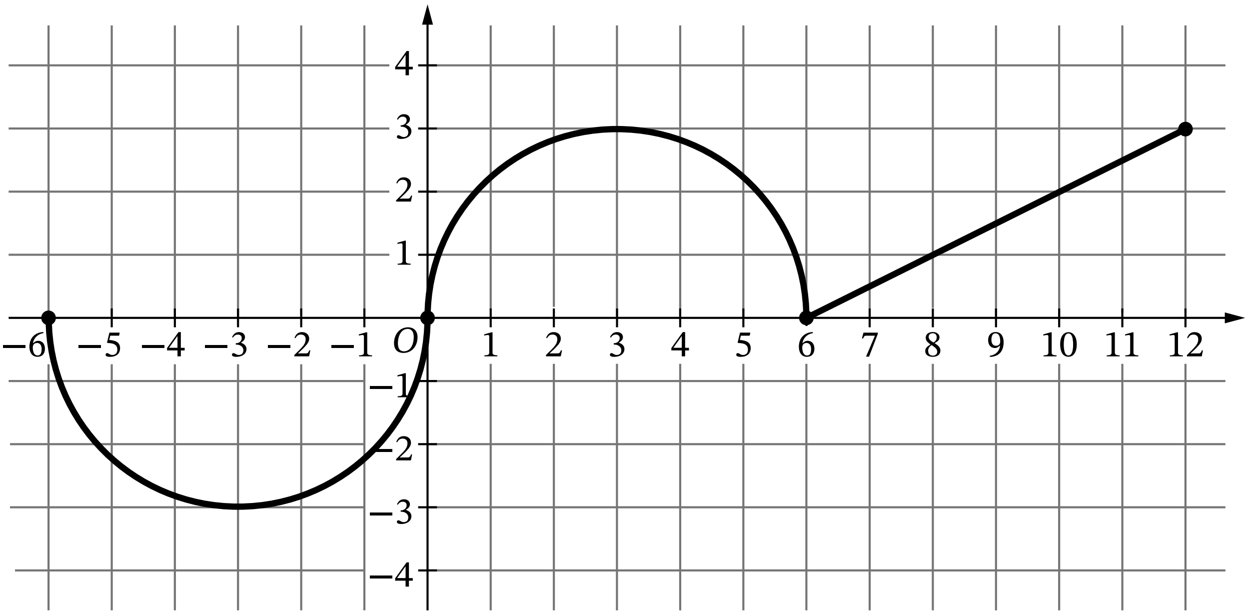
$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

4. The continuous function f is defined on the closed interval $-6 \le x \le 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.

A. Find $g'(8)$. Give a reason for your answer.

B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.

C. Find $g(12)$ and $g(0)$. Label your answers.

D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \le x \le 12$. Justify your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

   (c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

   (d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

2. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t seconds, where $x(t)$ and $y(t)$ are measured in centimeters. It is known that $x'(t) = 8t - t^2$ and $y'(t) = -t + \sqrt{t^{1.2} + 20}$. At time $t = 2$ seconds, the particle is at the point $(3, 6)$.

(a) Find the speed of the particle at time $t = 2$ seconds. Show the setup for your calculations.

(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 2$. Show the setup for your calculations.

(c) Find the y-coordinate of the position of the particle at the time $t = 0$. Show the setup for your calculations.

(d) For $2 \le t \le 8$, the particle remains in the first quadrant. Find all times t in the interval $2 \le t \le 8$ when the particle is moving toward the x-axis. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

END OF PART A

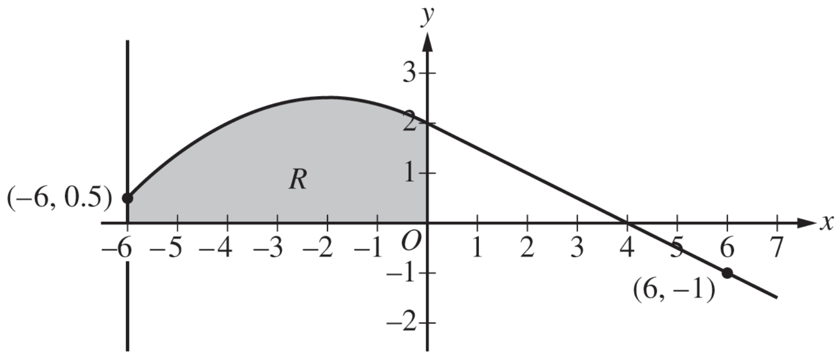
CALCULUS BC

SECTION II, Part B

Time—1 hour

4 Questions

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

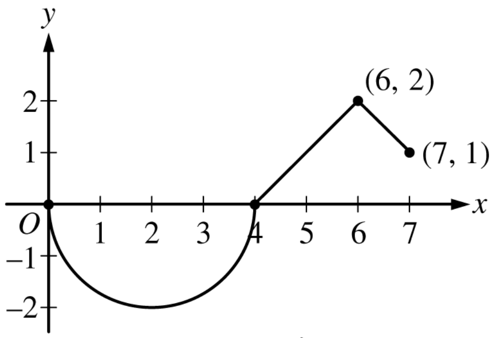
4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^x f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

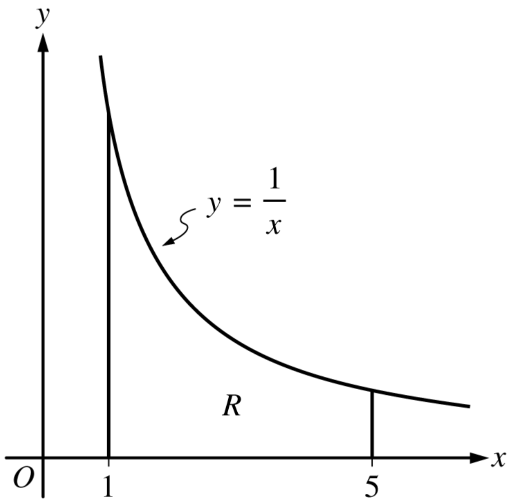


Figure 1

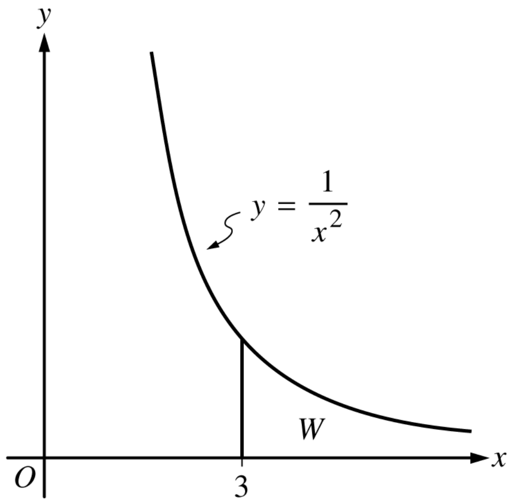


Figure 2

5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

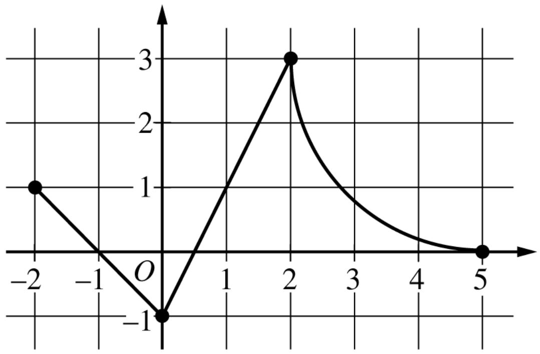
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

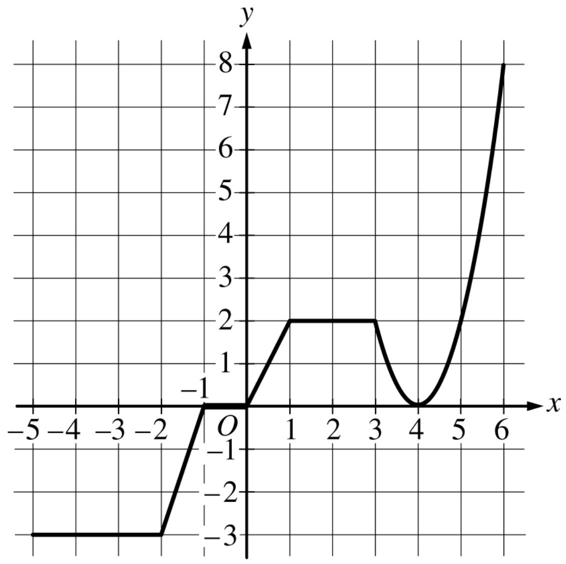
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x-4)^2$ for $3 \leq x \leq 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

   (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

   (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

   (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

   (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

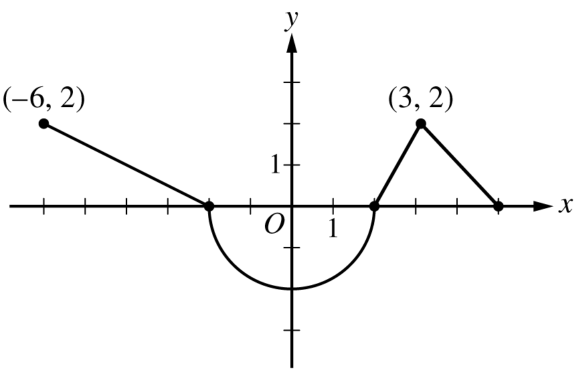
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

   (a) Find the values of $f(-6)$ and $f(5)$.

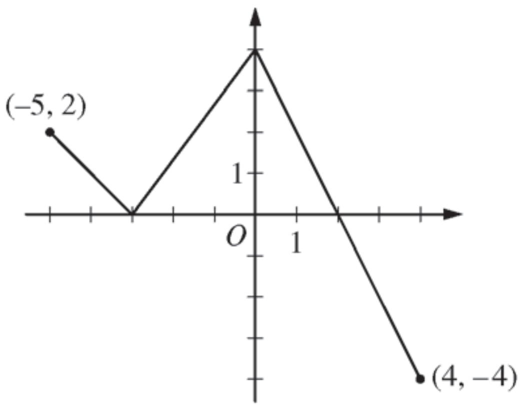
   (b) On what intervals is f increasing? Justify your answer.

   (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

   (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

6.8 Finding Antiderivatives and Indefinite Integrals

Name: ______ Class: ______ Date: ______ **Total: 13 marks**

KEY WORDS indefinite integral 不定积分 • constant of integration 积分常数 • antiderivative 原函数 • definite integral 定积分 • integral 积分

Objective

Build the skills to answer exam questions on **antiderivatives and indefinite integrals** — reversing differentiation.

You must be able to:

- apply the **power rule for integration** $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- integrate $\dfrac{1}{x}$, e^x, and basic trig functions
- always include the **constant of integration 积分常数** C

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The power rule for integration

Add one to the power and divide by the new power:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$$

So $\int x^3\,dx = \dfrac{x^4}{4} + C$ and $\int 5\,dx = 5x + C$.

■ The special case n = -1

The power rule fails for $n = -1$; instead $\int \dfrac{1}{x}\,dx = \ln|x| + C$.

■ **Standard antiderivatives**

$\int e^x\,dx = e^x + C$, $\int \cos x\,dx = \sin x + C$, $\int \sin x\,dx = -\cos x + C$, $\int \sec^2 x\,dx = \tan x + C$.

■ **Term by term (and never forget +C)**

Integrate a sum piece by piece: $\int (6x^2 - 4x + 1)\,dx = 2x^3 - 2x^2 + x + C$. The $+C$ is essential — an indefinite integral is a **family** of functions.

2 Practice

Now apply the methods above.

2.1 Find $\int x^5\,dx$. [1]

2.2 Find $\int (4x^3 - 2x)\,dx$. [2]

2.3 Find $\int \left(\frac{1}{x} + e^x\right)\,dx$. [2]

3 Exam-style questions

3.1 $\int \cos x \, dx =$ [1]

- **A** $-\sin x + C$
- **B** $\sin x + C$
- **C** $-\cos x + C$
- **D** $\tan x + C$

3.2 (a) Find the general antiderivative of $f(x) = 3x^2 + 4x$. [2]

(b) Find the particular antiderivative F with $F(0) = 5$. [2]

3.3 Find $\int \left(\sqrt{x} + \frac{2}{x}\right) dx$. Write $\sqrt{x}$ as $x^{1/2}$ first. [3]

Solutions

2.1 $\dfrac{x^6}{6} + C.$

2.2 $x^4 - x^2 + C.$

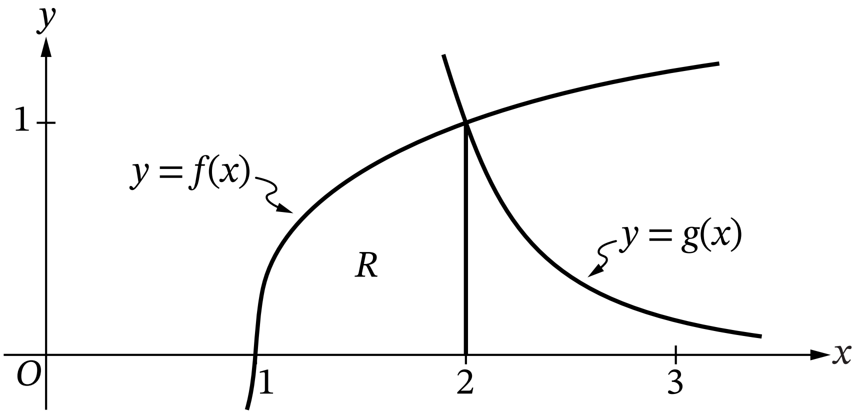
2.3 $\ln|x| + e^x + C.$

3.1 **B** — $\int \cos x\,dx = \sin x + C.$

3.2 (a) $F(x) = x^3 + 2x^2 + C$. (b) $F(0) = C = 5$, so $F(x) = x^3 + 2x^2 + 5$.

3.3 $\displaystyle\int (x^{1/2} + 2x^{-1})\,dx = \frac{x^{3/2}}{3/2} + 2\ln|x| + C = \tfrac{2}{3}x^{3/2} + 2\ln|x| + C.$

5. Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)dx$. Show the work that leads to your answer.

Part B

Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

Part D

Evaluate $\int_2^\infty g(x)dx$. Show the work that leads to your answer.

AP CALCULUS BC • EXERCISE SHEET

6.9 Integrating Using Substitution

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS integral 积分 • definite integral 定积分

Objective

Build the skills to answer exam questions on **integration by substitution** (u-substitution) — reversing the chain rule.

You must be able to:

- choose $u = g(x)$ so that its derivative $g'(x)$ also appears in the integrand
- convert dx to du and integrate in terms of u
- handle a **definite** integral by changing the limits or converting back to x

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Choosing u

Look for an "inside function" whose derivative is also present. For $\int 2x\,(x^2+1)^3\,dx$, let $u = x^2+1$, so $du = 2x\,dx$. The integral becomes

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C.$$

■ Adjusting for a constant

If the derivative is off by a constant, fix it. For $\int x\,(x^2+1)^3\,dx$, let $u = x^2+1$, $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$:

$$\int u^3 \cdot \tfrac{1}{2}\,du = \frac{u^4}{8} + C.$$

■ Definite integral — change the limits

For $\int_0^2 2x(x^2+1)^3\,dx$, with $u = x^2+1$: when $x=0$, $u=1$; when $x=2$, $u=5$. So

$$\int_1^5 u^3\,du = \left[\tfrac{u^4}{4}\right]_1^5 = \frac{625-1}{4} = 156.$$

> ■ **A trig example**
>
> $\int \cos(3x)\,dx$: let $u = 3x$, $du = 3\,dx$, so $dx = \frac{1}{3}\,du$, giving $\frac{1}{3}\sin(3x) + C$.

2 Practice

Now apply the methods above.

2.1 For $\int 3x^2(x^3+5)^4\,dx$, state a good choice of u and find du. [2]

2.2 Find $\int 2x\,e^{x^2}\,dx$. [2]

2.3 Find $\int \sin(4x)\,dx$. [2]

3 Exam-style questions

3.1 To integrate $\int x\cos(x^2)\,dx$, the best substitution is [1]

- **A** $u = x$
- **B** $u = \cos x$
- **C** $u = x^2$
- **D** $u = \cos(x^2)$

3.2 Find $\int (2x+1)(x^2+x)^5\,dx$. [3]

3.3 Evaluate $\int_0^1 3x^2\sqrt{x^3+1}\,dx$ using a substitution, changing the limits. [4]

Solutions

2.1 $u = x^3 + 5$; $du = 3x^2\,dx$.

2.2 $u = x^2$, $du = 2x\,dx$; $\int e^u\,du = e^{x^2} + C$.

2.3 $u = 4x$, $dx = \frac{1}{4}du$; $-\frac{1}{4}\cos(4x) + C$.

3.1 **C** — $u = x^2$ gives $du = 2x\,dx$, matching the x in the integrand.

3.2 $u = x^2 + x$, $du = (2x+1)\,dx$; $\int u^5\,du = \frac{u^6}{6} = \dfrac{(x^2+x)^6}{6} + C$.

3.3 $u = x^3 + 1$, $du = 3x^2\,dx$; limits $x = 0 \to u = 1$, $x = 1 \to u = 2$; $\int_1^2 \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^2$ $= \frac{2}{3}(2\sqrt{2} - 1) \approx 1.22$.

AP CALCULUS BC • EXERCISE SHEET

6.10 Integrating Using Long Division and Completing the Square

Name: ______ Class: ______ Date: ______ **Total: 14 marks**

KEY WORDS complete the square 配方 • polynomial long division 多项式长除法

Objective

Build the skills to answer exam questions on **integrating with long division and completing the square** — rewriting a rational integrand into an integrable form.

You must be able to:

- use **polynomial long division 多项式长除法** when the numerator's degree is $\geq$ the denominator's
- **complete the square 配方** in a denominator to reach an **arctangent** form
- recognise $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Long division first

$\displaystyle\int \frac{x^2}{x+1}\,dx$ is top-heavy. Divide: $\displaystyle\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Then

$$\int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

■ A simple 1/x-type

$\displaystyle\int \frac{1}{x-3}\,dx = \ln|x-3| + C$ (substitution $u = x-3$). Watch for these after dividing.

■ Completing the square

$\displaystyle\int \frac{1}{x^2+4x+5}\,dx$: complete the square, $x^2+4x+5 = (x+2)^2+1$. With $u = x+2$,

$$\int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+2) + C.$$

■ **The arctangent template**

$\displaystyle\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$. For $\displaystyle\int \frac{1}{x^2+9}\,dx = \tfrac{1}{3}\arctan\tfrac{x}{3} + C$.

2 Practice

Now apply the methods above.

2.1 Use long division to rewrite $\dfrac{x+3}{x+1}$ as $1 + \dfrac{?}{x+1}$. [2]

2.2 Find $\displaystyle\int \frac{1}{x^2+16}\,dx$. [2]

2.3 Complete the square: write $x^2 + 6x + 13$ in the form $(x+a)^2 + b$. [2]

3 Exam-style questions

3.1 $\displaystyle\int \frac{1}{x^2+25}\,dx =$ [1]

- **A** $\arctan(x) + C$
- **B** $\frac{1}{5}\arctan\frac{x}{5} + C$
- **C** $\ln(x^2+25) + C$
- **D** $5\arctan(5x) + C$

3.2 Find $\displaystyle\int \frac{x+2}{x+1}\,dx$. [3]

3.3 Find $\displaystyle\int \frac{1}{x^2+2x+10}\,dx$ by completing the square. [4]

Solutions

2.1 $\dfrac{x+3}{x+1} = 1 + \dfrac{2}{x+1}$.

2.2 $\frac{1}{4}\arctan\frac{x}{4} + C$.

2.3 $(x+3)^2 + 4$.

3.1 **B** — with $a = 5$, $\int \frac{dx}{x^2+25} = \frac{1}{5}\arctan\frac{x}{5} + C$.

3.2 $\dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$; $\int \left(1 + \frac{1}{x+1}\right) dx = x + \ln|x+1| + C$.

3.3 $x^2+2x+10 = (x+1)^2+9$; with $u = x+1$, $\int \frac{du}{u^2+9} = \frac{1}{3}\arctan\frac{u}{3} = \frac{1}{3}\arctan\dfrac{x+1}{3} + C$.

6.11 Integrating Using Integration by Parts

Name: ______________ Class: __________ Date: ____________ **Total: 16 marks**

KEY WORDS integration by parts 分部积分

Objective

Build the skills to answer exam questions on **integration by parts** — the reverse of the product rule.

You must be able to:

- apply $\displaystyle\int u\,dv = uv - \int v\,du$
- choose u and dv well (the **LIATE** guide)
- apply the method **twice** when needed

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The formula

Integration by parts reverses the product rule:

$$\int u\,dv = uv - \int v\,du.$$

Pick u (to differentiate) and dv (to integrate); then du and v follow.

Choosing u with LIATE

Choose u as the **first** type in **LIATE** that appears: **L**ogs, **I**nverse trig, **A**lgebraic, **T**rig, **E**xponential. For $\displaystyle\int xe^x\,dx$: x is algebraic (A), e^x exponential (E), so $u = x$, $dv = e^x\,dx$.

A worked example

$\displaystyle\int xe^x\,dx$: $u = x$, $dv = e^x\,dx \Rightarrow du = dx$, $v = e^x$:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

■ **Integrating a logarithm**

$\int \ln x\, dx$: let $u = \ln x$, $dv = dx$, so $du = \frac{1}{x}\, dx$, $v = x$:

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx = x \ln x - x + C.$$

2 Practice

Now apply the methods above.

2.1 For $\int x \cos x\, dx$, state a good choice of u and dv. [2]

2.2 Find $\int x \cos x\, dx$. [3]

2.3 Find $\int x e^{2x}\, dx$. [3]

3 Exam-style questions

3.1 In $\int x \sin x\, dx$, the best choice by LIATE is [1]

- **A** $u = \sin x$, $dv = x\, dx$
- **B** $u = x$, $dv = \sin x\, dx$
- **C** $u = x \sin x$, $dv = dx$
- **D** $u = 1$, $dv = x \sin x\, dx$

3.2 Find $\int \ln x \, dx$. [3]

3.3 Evaluate $\int_0^1 xe^x \, dx$. [4]

Solutions

2.1 $u = x$ (algebraic), $dv = \cos x\,dx$.

2.2 $du = dx$, $v = \sin x$; $\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C$.

2.3 $u = x$, $dv = e^{2x}\,dx \Rightarrow v = \frac{1}{2}e^{2x}$; $\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

3.1 **B** — x is algebraic (A), $\sin x$ is trig (T); A comes before T, so $u = x$.

3.2 $u = \ln x$, $dv = dx$, $v = x$; $x \ln x - \int x \cdot \frac{1}{x}\,dx = x \ln x - x + C$.

3.3 $u = x$, $dv = e^x\,dx$; $\left[xe^x\right]_0^1 - \int_0^1 e^x\,dx = (e - 0) - \left[e^x\right]_0^1 = e - (e - 1) = 1$.

x	0	π	2π
$f'(x)$	5	6	0

5. The function f is twice differentiable for all x with $f(0) = 0$. Values of f', the derivative of f, are given in the table for selected values of x.

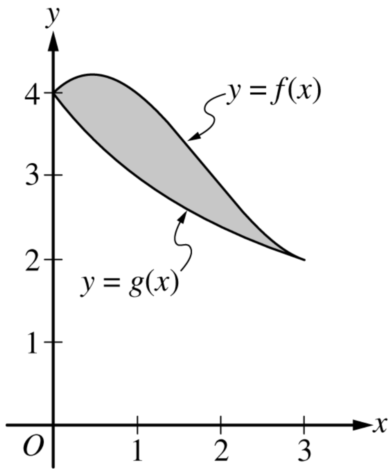
(a) For $x \geq 0$, the function h is defined by $h(x) = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$. Find the value of $h'(\pi)$. Show the work that leads to your answer.

(b) What information does $\int_0^\pi \sqrt{1 + (f'(x))^2}\, dx$ provide about the graph of f ?

(c) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(2\pi)$. Show the computations that lead to your answer.

(d) Find $\int (t + 5)\cos\left(\frac{t}{4}\right) dt$. Show the work that leads to your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3 + x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

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6.12 Integrating Using Linear Partial Fractions

Name: ______________ Class: __________ Date: ____________ **Total: 16 marks**

KEY WORDS partial fractions 部分分式

Objective

Build the skills to answer exam questions on **integration by linear partial fractions** — splitting a rational function to integrate it.

You must be able to:

- factor the denominator and write the **partial-fraction 部分分式** decomposition
- solve for the unknown constants
- integrate each simple fraction to a **logarithm**

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The decomposition

A proper fraction with a factored denominator splits into simpler pieces:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Solving for A and B

Multiply through by $(x-1)(x+2)$: $1 = A(x+2) + B(x-1)$. Set $x = 1$: $1 = 3A \Rightarrow A = \frac{1}{3}$. Set $x = -2$: $1 = -3B \Rightarrow B = -\frac{1}{3}$.

Integrating the pieces

Each piece integrates to a log:

$$\int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2} \right) dx = \tfrac{1}{3} \ln|x-1| - \tfrac{1}{3} \ln|x+2| + C.$$

■ **When to divide first**

If the fraction is **improper** (numerator degree $\geq$ denominator), do **long division** first, then apply partial fractions to the remainder.

2 Practice

Now apply the methods above.

2.1 Write the partial-fraction form of $\dfrac{1}{(x-3)(x+1)}$ (with unknowns A, B). [1]

2.2 Find A and B for $\dfrac{5}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3}$. [3]

2.3 Integrate $\displaystyle\int \left(\frac{2}{x-1} + \frac{3}{x+4}\right) dx$. [2]

3 Exam-style questions

3.1 After decomposing, $\displaystyle\int \frac{A}{x-a}\, dx =$ [1]

- **A** $A \ln|x-a| + C$
- **B** $\dfrac{A}{(x-a)^2} + C$
- **C** $A(x-a) + C$
- **D** $\dfrac{A}{2}(x-a)^2 + C$

3.2 Decompose and integrate $\displaystyle\int \frac{1}{(x-1)(x+1)}\, dx$.

(a) Find the constants. [2]

(b) Complete the integration. [2]

3.3 Find $\displaystyle\int \frac{x+7}{(x-1)(x+3)}\,dx$. [5]

Solutions

2.1 $\dfrac{A}{x-3} + \dfrac{B}{x+1}$.

2.2 $5 = A(x+3) + B(x-2)$; $x = 2$: $5 = 5A \Rightarrow A = 1$; $x = -3$: $5 = -5B \Rightarrow B = -1$.

2.3 $2\ln|x-1| + 3\ln|x+4| + C$.

3.1 **A** — $\int \frac{A}{x-a}\,dx = A\ln|x-a| + C$.

3.2 (a) $1 = A(x+1) + B(x-1)$; $x = 1$: $A = \frac{1}{2}$; $x = -1$: $B = -\frac{1}{2}$. (b) $\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$.

3.3 $x+7 = A(x+3) + B(x-1)$; $x = 1$: $8 = 4A \Rightarrow A = 2$; $x = -3$: $4 = -4B \Rightarrow B = -1$; $\int \left(\frac{2}{x-1} - \frac{1}{x+3}\right)dx = 2\ln|x-1| - \ln|x+3| + C$.

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\displaystyle\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

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5. Consider the function $f(x) = \dfrac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

(a) Let $k = 3$, so that $f(x) = \dfrac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

(b) Let $k = 4$, so that $f(x) = \dfrac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

(c) Find the value of k for which f has a critical point at $x = -5$.

(d) Let $k = 6$, so that $f(x) = \dfrac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\displaystyle\int f(x)\,dx$.

6.13 Evaluating Improper Integrals

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS diverges 发散 • improper integral 反常积分 • limit 极限 • converges 收敛 • integral 积分

Objective

Build the skills to answer exam questions on **improper integrals** — integrals with an infinite limit or an infinite discontinuity.

You must be able to:

- rewrite an improper integral as a **limit** 极限
- evaluate the limit to decide **convergence** 收敛 or **divergence** 发散
- handle a discontinuity inside the interval

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Infinite upper limit

Replace ∞ with b and take a limit:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx = \lim_{b\to\infty}\Big[-\tfrac{1}{x}\Big]_1^b = \lim_{b\to\infty}\Big(-\tfrac{1}{b}+1\Big) = 1.$$

The limit is finite, so the integral **converges** to 1.

■ A divergent example

$\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\big[\ln x\big]_1^b = \lim_{b\to\infty}\ln b = \infty.$ The limit is infinite, so it **diverges**.

■ Discontinuity at an endpoint

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is improper at $x=0$: $\lim_{a\to0^+}\int_a^1 x^{-1/2}\,dx = \lim_{a\to0^+}\big[2\sqrt{x}\big]_a^1 = 2-0 = 2.$ Converges.

■ The rule for 1/x^p on [1, ∞)

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ **converges** when $p>1$ and **diverges** when $p\le 1$ — a handy check.

2 Practice

Now apply the methods above.

2.1 Rewrite $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ as a limit. [1]

2.2 Evaluate $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx.$ [3]

2.3 State whether $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,dx$ converges or diverges (use the p-rule). [1]

3 Exam-style questions

3.1 $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges when [1]

- **A** $p < 1$
- **B** $p = 1$
- **C** $p > 1$
- **D** all p

3.2 Evaluate $\displaystyle\int_0^\infty e^{-x}\,dx.$

(a) Write it as a limit. [1]

(b) Evaluate. [3]

3.3 Determine whether $\int_2^{\infty} \frac{1}{x}\, dx$ converges or diverges, showing your reasoning with a limit. [3]

Solutions

2.1 $\lim_{b\to\infty} \int_1^b x^{-3}\,dx.$

2.2 $\left[-\frac{1}{2x^2}\right]_1^b = -\frac{1}{2b^2} + \frac{1}{2}$; as $b \to \infty$ this $\to \frac{1}{2}$.

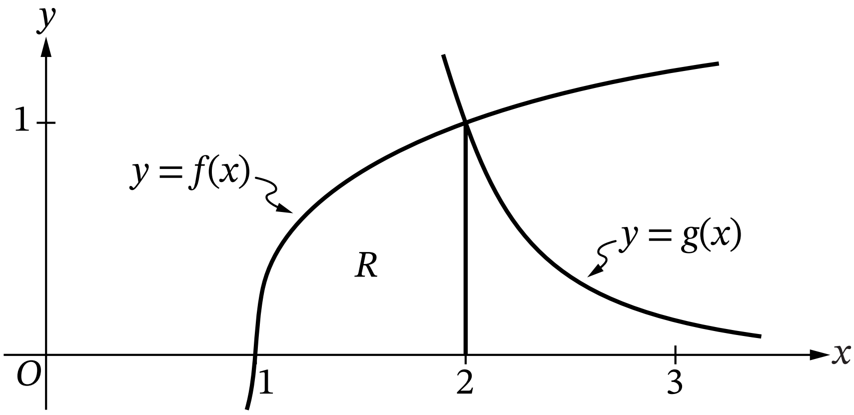
2.3 Diverges — $p = \frac{1}{2} \le 1$.

3.1 **C** — converges for $p > 1$.

3.2 (a) $\lim_{b\to\infty} \int_0^b e^{-x}\,dx$. (b) $\left[-e^{-x}\right]_0^b = -e^{-b} + 1$; $\to 1$ as $b \to \infty$.

3.3 $\lim_{b\to\infty} \left[\ln x\right]_2^b = \lim_{b\to\infty} (\ln b - \ln 2)$; $\ln b \to \infty$, so it **diverges**.

5. Let f be the function defined by $f(x) = \sqrt[3]{x-1}$, and let g be the function defined by $g(x) = e^{(-2x+4)}$. The graphs of f and g intersect at the point $(2, 1)$. Let R be the region bounded by the graph of f, the x-axis, and the vertical line $x = 2$, as shown in the figure.



Part A

Evaluate $\int_1^2 f(x)dx$. Show the work that leads to your answer.

Part B

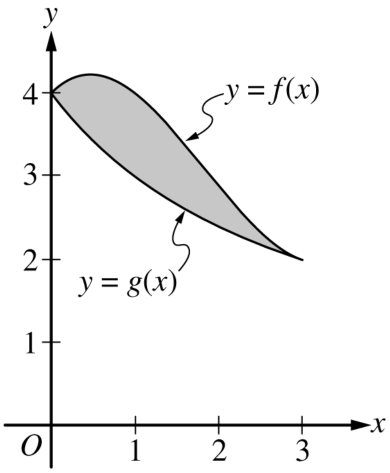
Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid generated when region R is rotated about the x-axis.

Part C

Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of region R.

Part D

Evaluate $\int_2^\infty g(x)dx$. Show the work that leads to your answer.



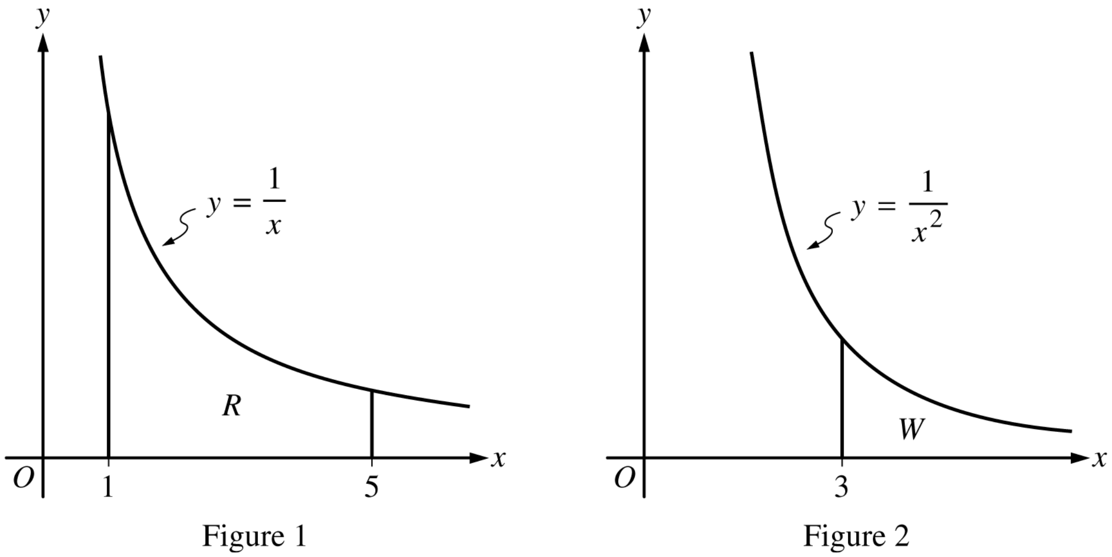
5. The graphs of the functions f and g are shown in the figure for $0 \le x \le 3$. It is known that $g(x) = \dfrac{12}{3+x}$ for $x \ge 0$. The twice-differentiable function f, which is not explicitly given, satisfies $f(3) = 2$ and $\int_0^3 f(x)\,dx = 10$.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Evaluate the improper integral $\int_0^\infty (g(x))^2\,dx$, or show that the integral diverges.

(c) Let h be the function defined by $h(x) = x \cdot f'(x)$. Find the value of $\int_0^3 h(x)\,dx$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \dfrac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \dfrac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\, dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\, dx$ or show that it diverges.

2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \le h \le 30$ and is modeled by $f(h)$ for $h \ge 30$. The continuous function f is not explicitly given.

   (a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

   (b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

   (c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\, dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

   (d) The boat is moving on the surface of the sea. At time $t \ge 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \le t \le 1$.

END OF PART A OF SECTION II

5. Let f be the function defined by $f(x) = \dfrac{\text{[illegible]}}{2x^2 - 7x + 5}$.

   (a) Find the slope of the line tangent to the graph of f at $x = 3$.

   (b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

   (c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^{\infty} f(x)\, dx$ or show that the integral diverges.

   (d) Determine whether the series $\sum_{n=5}^{\infty} \dfrac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

6.14 Selecting Techniques for Antidifferentiation

Name: ______________ Class: __________ Date: ____________ **Total: 10 marks**

KEY WORDS antiderivative 原函数 • integral 积分

Objective

Build the skills to answer exam questions on **selecting antidifferentiation techniques** — choosing the right method quickly.

You must be able to:

- recognise when a **basic rule**, a **substitution**, **long division**, or **completing the square** applies
- match an integrand's shape to the fastest method
- carry the chosen method through to a correct antiderivative

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The decision guide

- **Basic rule** — a power, e^x, sin / cos, or $\frac{1}{x}$ on its own.
- **Substitution** — an inside function whose derivative also appears (e.g. $2x$ beside $x^2 + 1$).
- **Long division** — a rational function with numerator degree $\geq$ denominator degree.
- **Completing the square** — an irreducible quadratic x^2+bx+c in a denominator (arctangent).

Spotting a substitution

$\int x^2 e^{x^3}\,dx$: the x^2 is (up to a constant) the derivative of x^3 — **substitute** $u = x^3$. Answer $\frac{1}{3}e^{x^3} + C$.

■ **Spotting a division**

$\int \frac{x^3}{x^2+1}\,dx$: top-heavy, so **divide** first: $x - \frac{x}{x^2+1}$, then integrate (the second piece needs $u = x^2+1$).

■ **Spotting a basic rule**

$\int (2x + \cos x)\,dx$ needs no tricks — just the **basic rules**: $x^2 + \sin x + C$.

2 Practice

Now apply the methods above. For each, name the best method (basic / substitution / division / complete-the-square).

2.1 $\int \frac{1}{x^2+4}\,dx$ [1]

2.2 $\int 5x^4(x^5+2)^7\,dx$ [1]

2.3 $\int \frac{x^2+1}{x}\,dx$ [1]

3 Exam-style questions

3.1 Which integral is best done by **substitution**? [1]

- **A** $\int (x^3 - 2)\, dx$
- **B** $\int \frac{1}{x^2 + 9}\, dx$
- **C** $\int \cos(x^2) \cdot 2x\, dx$
- **D** $\int \frac{x^2}{x + 1}\, dx$

3.2 Find $\int \frac{x^2 + 3}{x}\, dx$ (rewrite the fraction first). [3]

3.3 Find $\int \frac{6x}{x^2 + 1}\, dx$. [3]

Solutions

2.1 Complete-the-square / arctangent form (basic arctangent rule).

2.2 Substitution, $u = x^5 + 2$.

2.3 Basic — split into $x + \frac{1}{x}$ first.

3.1 **C** — $2x$ is the derivative of x^2, so substitute $u = x^2$.

3.2 $\dfrac{x^2+3}{x} = x + \dfrac{3}{x}$; $\int \left(x + \frac{3}{x}\right) dx = \frac{x^2}{2} + 3\ln|x| + C$.

3.3 $u = x^2 + 1$, $du = 2x\,dx$, so $6x\,dx = 3\,du$; $\int \frac{3}{u}\,du = 3\ln|u| = 3\ln(x^2+1) + C$.