

Week 12

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

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AP Calculus BC

Name: _____ Score: _____

1. The MVT on $[a, b]$ requires **which** hypotheses?

- ☐ f continuous on $[a, b]$
- ☐ f differentiable on (a, b)
- ☐ $f(a) = f(b)$
- ☐ f is a polynomial

Tick every correct option.

2. A critical point is an interior x where...

- ☐ $f(x) = 0$
- ☐ $f'(x) = 0$ or $f'(x)$ does not exist
- ☐ $f(x)$ is largest
- ☐ $f''(x) = 0$ only

3. A function is increasing on an interval exactly when...

- ☐ $f(x) > 0$ there
- ☐ $f'(x) > 0$ there
- ☐ $f(x) < 0$ there
- ☐ $f''(x) > 0$ there

4. If f' changes from positive to negative at c , then f has a...

- ☐ local minimum
- ☐ local maximum
- ☐ inflection point
- ☐ vertical asymptote

5. To find absolute extrema on $[a, b]$, evaluate f at **which** points?

- ☐ all critical points in $[a, b]$
- ☐ the endpoint a
- ☐ the endpoint b
- ☐ every integer in $[a, b]$

Tick every correct option.

6. A graph is concave up exactly where...

☐ $f'(x) > 0$

☐ $f''(x) > 0$

☐ $f(x) > 0$

☐ $f''(x) < 0$

7. If $f'(c) = 0$ and $f''(c) > 0$, then c is a...

☐ relative maximum

☐ relative minimum

☐ inflection point

☐ endpoint

8. For $A(x) = 40x - 2x^2$, solve $A'(x) = 40 - 4x = 0$ for the critical x .

9. For $f(x) = x^2$ on $[2, 6]$, find the c guaranteed by the MVT.

10. Find **all** critical points of $f(x) = x^3 - 3x$.

☐ $x = 1$

☐ $x = -1$

☐ $x = 0$

☐ $x = 3$

Tick every correct option.

AP Calculus BC1. **f continuous on $[a, b]$; f differentiable on (a, b)**

Continuous on the closed interval and differentiable on the open one. ($f(a) = f(b)$ is the extra Rolle condition.)

2. **$f'(x) = 0$ or $f'(x)$ does not exist**

Critical points: $f' = 0$ or f' undefined.

3. **$f'(x) > 0$ there**

Positive first derivative means increasing.

4. **local maximum**

Up then down = the top of a hill = local max.

5. **all critical points in $[a, b]$; the endpoint a ; the endpoint b**

Critical points plus both endpoints — that is the full candidate list.

6. **$f''(x) > 0$**

Concave up $\Leftrightarrow f'' > 0$ (slope increasing).

7. **relative minimum**

Concave up at a flat spot = minimum.

8. **10**

$$40 - 4x = 0 \Rightarrow x = 10.$$

9. **4**

$$\text{Average} = \frac{36-4}{4} = 8; 2c = 8 \Rightarrow c = 4 \in (2, 6).$$

10. **$x = 1$; $x = -1$**

$$f'(x) = 3(x-1)(x+1) = 0 \text{ at } x = \pm 1.$$

5.1 Using the Mean Value Theorem

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS mean value theorem 中值定理 • increasing 递增

Objective

Build the skills to answer exam questions on **the Mean Value Theorem**.

You must be able to:

- state the **Mean Value Theorem** 中值定理 and its hypotheses
- find the guaranteed point c

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ The Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) , there is a c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

— somewhere the tangent slope equals the average slope.

■ Example

$f(x) = x^2$ on $[0, 4]$: average slope $= \frac{16-0}{4} = 4$; $f'(c) = 2c = 4$ gives $c = 2$.

2 Practice

2.1 State the MVT conclusion. [1]

2.2 State the two hypotheses. [1]

2.3 For $f(x) = x^2$ on $[0, 4]$, find the average rate of change. [2]

3 Exam-style questions

3.1 The MVT guarantees a c with $f'(c)$ equal to [1]

- **A** 0
 - **B** the average rate of change on $[a, b]$
 - **C** $f(a)$
 - **D** ∞
-

3.2 The MVT requires f continuous on $[a, b]$ and [1]

- **A** $f = 0$
 - **B** differentiable on (a, b)
 - **C** linear
 - **D** increasing
-

3.3 $f(x) = x^2$ on $[0, 4]$.

(a) Find $f(0)$ and $f(4)$. [1]

(b) Find the average slope. [1]

(c) Solve $f'(c) =$ that for c .

[1]

Solutions

2.1 there is a c in (a, b) where $f'(c)$ equals the average rate of change on $[a, b]$.

2.2 f continuous on $[a, b]$ and differentiable on (a, b) .

2.3 $\frac{16 - 0}{4 - 0} = 4.$

3.1 B.

3.2 B.

3.3 (a) $f(0) = 0$, $f(4) = 16$. (b) 4. (c) $2c = 4$, so $c = 2$.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f , where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

- (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t) dt$ in the context of the problem. Use a right

Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of

$$\int_{60}^{135} f(t) dt.$$

- (b) Must there exist a value of c , for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

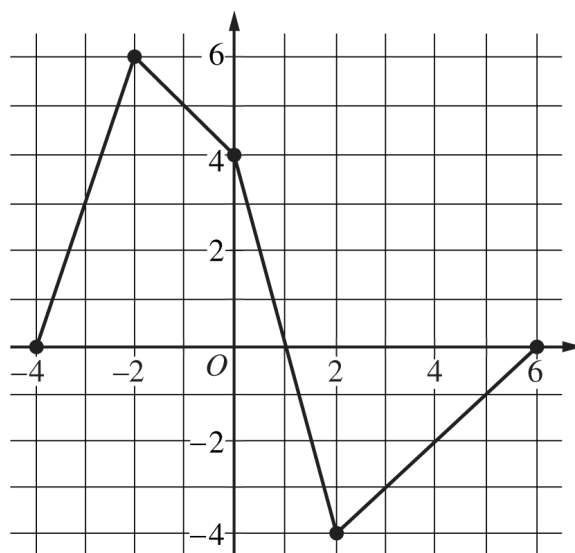
- (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right)$ for

$0 \leq t \leq 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \leq t \leq 150$.

Show the setup for your calculations.

- (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Graph of f

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f , consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t) \, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \rightarrow 2} \frac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c , $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?
-

5.2 Extreme Values, Global vs Local Extrema, and Critical Points

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS critical point 临界点 • extreme value theorem 极值定理

Objective

Build the skills to answer exam questions on the **Extreme Value Theorem** and **critical points**.

You must be able to:

- state the **Extreme Value Theorem** 极值定理
- find the **critical points** 临界点 of a function

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ EVT and critical points

Extreme Value Theorem: a function continuous on a closed interval $[a, b]$ attains a global maximum and minimum.

Critical points are where $f'(x) = 0$ or f' does not exist — the candidates for local extrema.

■ Example

$f(x) = x^2 - 4x$: $f'(x) = 2x - 4 = 0$ gives the critical point $x = 2$.

2 Practice

2.1 State the Extreme Value Theorem. [1]

2.2 State what a critical point is. [1]

2.3 Find the critical point of $f(x) = x^2 - 4x$. [2]

3 Exam-style questions

3.1 A critical point occurs where [1]

- **A** $f = 0$
 - **B** $f' = 0$ or f' does not exist
 - **C** $f'' = 0$
 - **D** $x = 0$
-

3.2 The EVT guarantees extrema when f is continuous on [1]

- **A** an open interval
 - **B** a closed interval
 - **C** all real numbers
 - **D** no interval
-

3.3 $f(x) = x^2 - 6x$.

(a) Find $f'(x)$. [1]

(b) Set $f' = 0$. [1]

(c) State the critical point.

[1]

Solutions

2.1 a function continuous on a closed interval attains a global max and min.

2.2 a point where $f' = 0$ or f' does not exist.

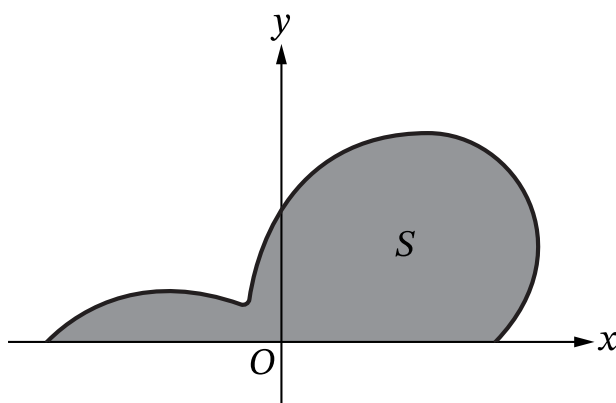
2.3 $f'(x) = 2x - 4 = 0$, so $x = 2$.

3.1 B.

3.2 B.

3.3 (a) $2x - 6$. (b) $2x - 6 = 0$. (c) $x = 3$.

2. Let S be the shaded region bounded by the graph of the polar curve $r(\theta) = 3 + 2 \sin(2\theta) + \cos(2\theta)$ for $0 \leq \theta \leq \pi$, as shown in the figure.



It can be shown that $r'(\theta) = 4 \cos(2\theta) - 2 \sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Part A

Find the area of region S . Show the setup for your calculations.

Part B

There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Part C

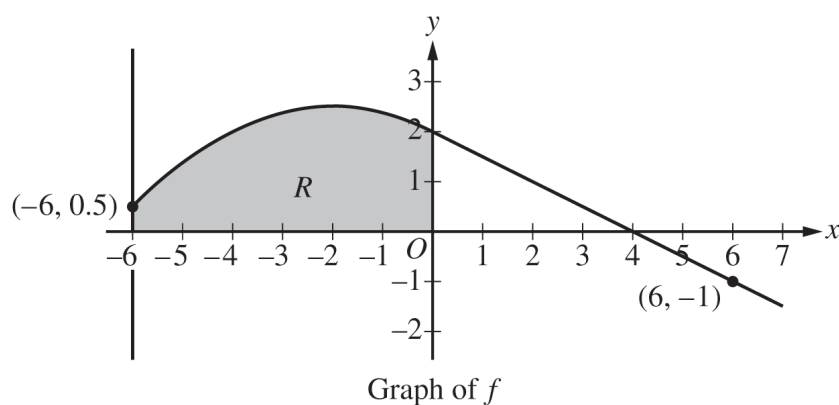
- Find the value of θ in the interval $0 < \theta < \frac{\pi}{2}$ at which r has a critical point.
- Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r .

Part D

Find the average distance from the origin to a point on the polar curve

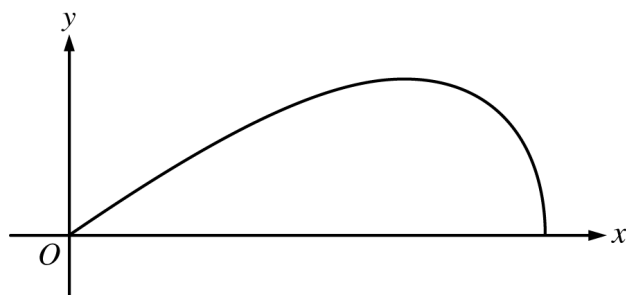
$r(\theta) = 3 + 2 \sin(2\theta) + \cos(2\theta)$ for $\frac{\pi}{2} \leq \theta \leq \pi$. Show the setup for your calculations.

END OF PART A



4. The graph of the differentiable function f , shown for $-6 \leq x \leq 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \leq x \leq 7$. Let R be the region in the second quadrant bounded by the graph of f , the vertical line $x = -6$, and the x - and y -axes. Region R has area 12.
- (a) The function g is defined by $g(x) = \int_0^x f(t) \, dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.
- (b) For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.
- (c) The function h is defined by $h(x) = \int_{-6}^x f'(t) \, dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x -axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c . Each spinning toy is in the shape of the solid generated when such a region is revolved about the x -axis. Both x and y are measured in inches.

- (a) Find the area of the region in the first quadrant bounded by the x -axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.
- (b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?
- (c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.

- (a) Find the slope of the line tangent to the graph of f at $x = 3$.
- (b) Find the x -coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.
- (c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^\infty f(x) \, dx$ or show that the integral diverges.
- (d) Determine whether the series $\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x -coordinate is 4.
- (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.
- (c) Find the value of k for which f has a critical point at $x = -5$.
- (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f .
Find $\int f(x) \, dx$.

5.3 Where a Function Increases or Decreases

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS increasing 递增 • decreasing 递减 • critical point 临界点

Objective

Build the skills to answer exam questions on **intervals where a function is increasing or decreasing**.

You must be able to:

- use the sign of f' to find where f is **increasing** 递增 or **decreasing** 递减

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Increasing and decreasing

- f is **increasing** where $f' > 0$;
- f is **decreasing** where $f' < 0$.

Test the sign of f' on the intervals between critical points.

■ Example

$f'(x) = 2x - 4$: $f' > 0$ for $x > 2$ (increasing) and $f' < 0$ for $x < 2$ (decreasing).

2 Practice

2.1 State where f is increasing. [1]

2.2 For $f'(x) = 2x - 4$, find where f is increasing. [2]

2.3 For the same f' , state where f is decreasing. [1]

3 Exam-style questions

3.1 f is increasing where [1]

- **A** $f' < 0$
 - **B** $f' > 0$
 - **C** $f'' > 0$
 - **D** $f = 0$
-

3.2 If $f'(x) = 2x - 4$, then f is decreasing for [1]

- **A** $x > 2$
 - **B** $x < 2$
 - **C** all x
 - **D** $x = 2$
-

3.3 $f'(x) = 3x^2 - 3$.

(a) Solve $f' = 0$. [1]

(b) Test a point in $(-1, 1)$. [1]

(c) State where f is decreasing.

[1]

Solutions

2.1 where $f' > 0$.

2.2 $2x - 4 > 0 \Rightarrow x > 2$.

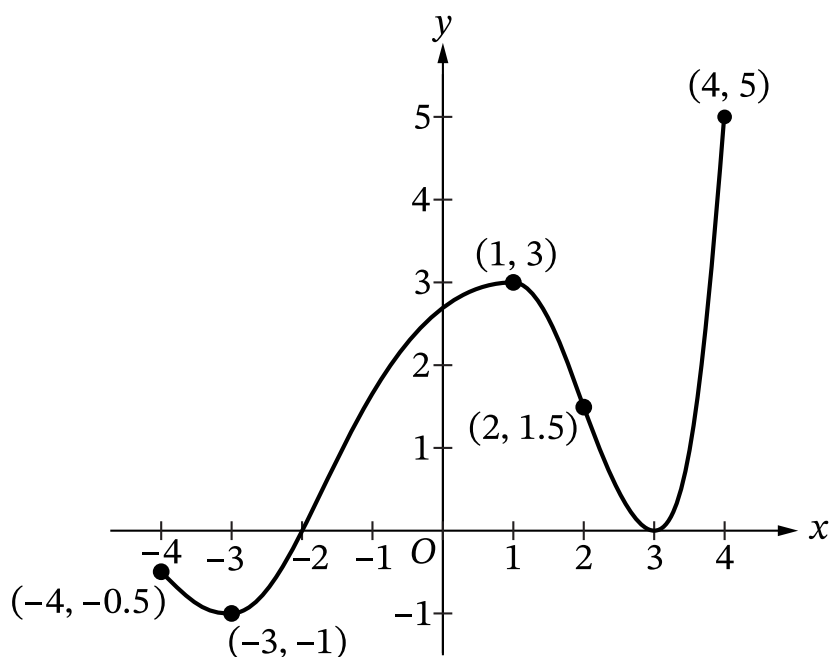
2.3 $x < 2$.

3.1 B.

3.2 B.

3.3 (a) $x = \pm 1$. (b) $f'(0) = -3 < 0$. (c) $(-1, 1)$.

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f' , the derivative of f , is shown.

Graph of f' **Part A**

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Part C

For $-4 \leq x \leq 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

Part D

For $-4 \leq x \leq 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

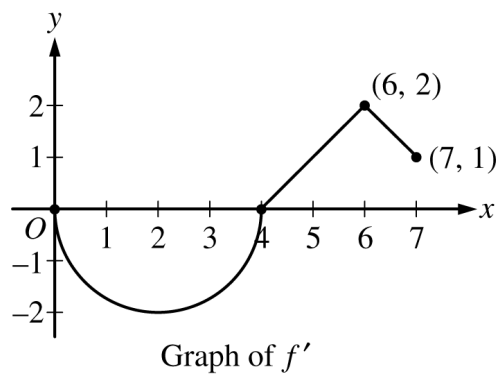
1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by

$A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour.

Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

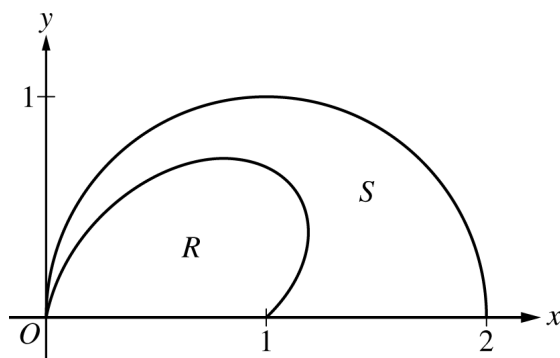
- (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).
- (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).
- (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.
- (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t , for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400) dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f' , the derivative of f , consists of a semicircle and two line segments, as shown in the figure above.
- (a) Find $f(0)$ and $f(5)$.
- (b) Find the x -coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.
- (c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
- (d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin \theta \cos(2\theta)$ and $r = g(\theta) = 2 \cos \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x -axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x -axis.
- (a) Find the area of R .
- (b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k .
- (c) For each θ , $0 \leq \theta \leq \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A , the average value of $w(\theta)$ over the interval $0 \leq \theta \leq \frac{\pi}{2}$.
- (d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

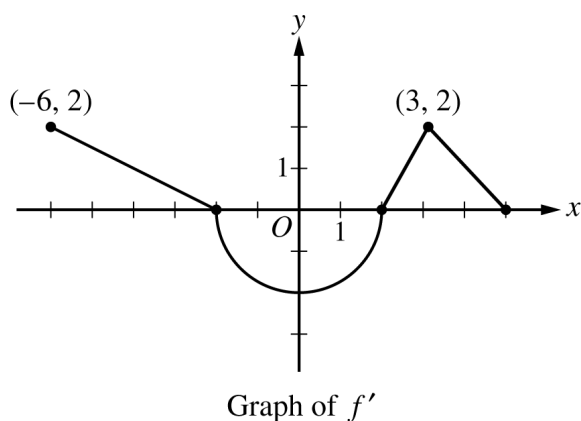
END OF PART A OF SECTION II

CALCULUS BC
SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

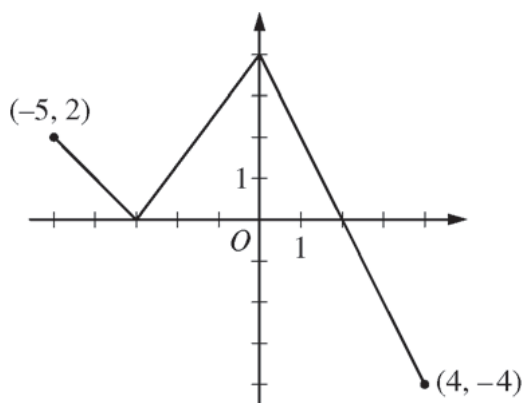


3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f' , the derivative of f , consists of a semicircle and three line segments, as shown in the figure above.
- (a) Find the values of $f(-6)$ and $f(5)$.
- (b) On what intervals is f increasing? Justify your answer.
- (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.
- (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.
-

CALCULUS BC
SECTION II, Part A**Time—30 minutes****Number of problems—2****A graphing calculator is required for these problems.**

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \leq t \leq 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \leq t \leq 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.
- (a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.
- (b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.
- (c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.
- (d) For $0 \leq t \leq 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.
-

CALCULUS BC
SECTION II, Part B**Time—60 minutes****Number of problems—4****No calculator is allowed for these problems.**Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^x f(t) dt$.
- (a) Find $g(3)$.
- (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.
- (c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.
- (d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.
-

5.4 The First Derivative Test for Local Extrema

Name: _____ Class: _____ Date: _____

Total: 9 marks**KEY WORDS** local minimum 极小值 • local maximum 极大值 • local extremum 局部极值 • critical point 临界点

Objective

Build the skills to answer exam questions on **the First Derivative Test**.

You must be able to:

- use the sign change of f' to classify a **local extremum** 局部极值

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ First Derivative Test

At a critical point, look at how f' changes:

- $+$ to $-$ $\rightarrow$ local **maximum**;
- $-$ to $+$ $\rightarrow$ local **minimum**;
- no sign change $\rightarrow$ neither.

■ Example

$f'(x) = 2x - 4$ at $x = 2$: f' goes $-$ (for $x < 2$) to $+$ (for $x > 2$), so $x = 2$ is a local minimum.

2 Practice

2.1 State the sign change for a local maximum. [1]

2.2 State the sign change for a local minimum. [1]

2.3 For $f'(x) = 2x - 4$, classify $x = 2$. [2]

3 Exam-style questions

3.1 A local maximum occurs where f' changes [1]

- **A** $-$ to $+$
 - **B** $+$ to $-$
 - **C** stays $+$
 - **D** stays $-$
-

3.2 A local minimum occurs where f' changes [1]

- **A** $+$ to $-$
 - **B** $-$ to $+$
 - **C** stays 0
 - **D** never
-

3.3 $f'(x) = 2x - 4$, critical at $x = 2$.

(a) State the sign of f' for $x < 2$. [1]

(b) State the sign for $x > 2$. [1]

(c) Classify $x = 2$.

[1]

Solutions

2.1 f' changes from $+$ to $-$.

2.2 f' changes from $-$ to $+$.

2.3 f' goes $-$ to $+$, so $x = 2$ is a local minimum.

3.1 B.

3.2 B.

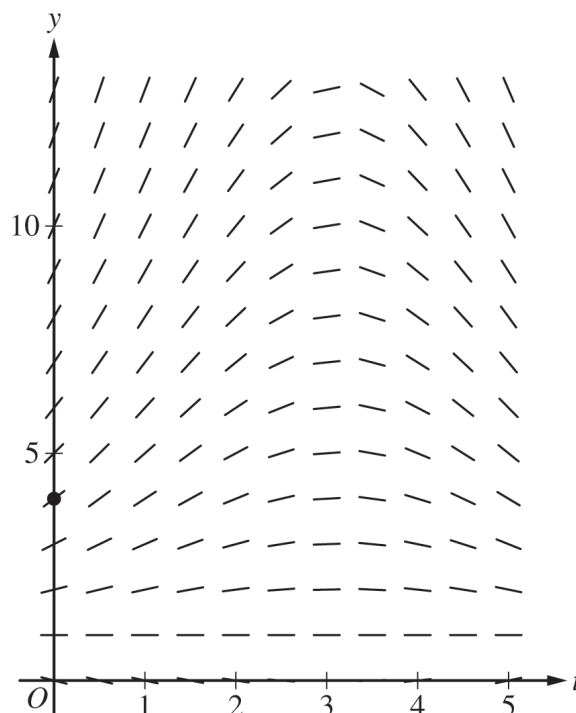
3.3 (a) negative. (b) positive. (c) local minimum.

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation

$$\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right), \text{ where } H(t) \text{ is measured in feet and } t \text{ is measured in hours after noon } (t = 0). \text{ It is}$$

known that $H(0) = 4$.

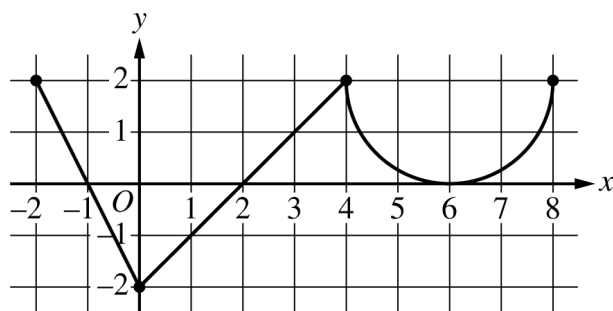
- (a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.



- (b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t , for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.
- (c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation

$$\frac{dH}{dt} = \frac{1}{2}(H - 1)\cos\left(\frac{t}{2}\right) \text{ with initial condition } H(0) = 4.$$

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Graph of f'

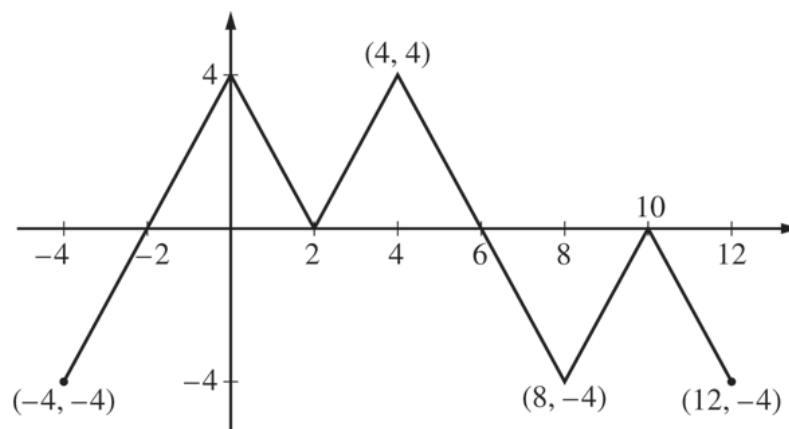
4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f' , the derivative of f , consists of two line segments and a semicircle, as shown in the figure.
- (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.
- (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.
- (c) Find the value of $\lim_{x \rightarrow 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.
- (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

1. From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.
- (a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).
- (b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).
- (c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.
- (d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t , for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400) dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Let f be the function defined by $f(x) = \frac{3}{2x^2 - 7x + 5}$.
- (a) Find the slope of the line tangent to the graph of f at $x = 3$.
- (b) Find the x -coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.
- (c) Using the identity that $\frac{3}{2x^2 - 7x + 5} = \frac{2}{2x - 5} - \frac{1}{x - 1}$, evaluate $\int_5^\infty f(x) dx$ or show that the integral diverges.
- (d) Determine whether the series $\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

CALCULUS BC
SECTION II, Part B**Time—60 minutes****Number of problems—4****No calculator is allowed for these problems.**Graph of f

3. The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t) dt$.

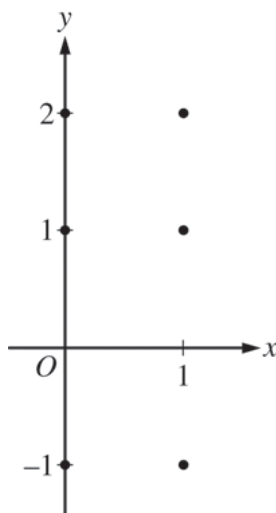
- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
 - (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
 - (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
 - (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.
-

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

- (a) Find $\frac{d^2y}{dx^2}$ in terms of x and y .
 - (b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.
 - (c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.
 - (d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.
-

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



- (b) Find $\frac{d^2y}{dx^2}$ in terms of x and y . Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.
- (c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.
- (d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.
-

5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}.$$

- (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x -coordinate is 4.
- (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.
- (c) Find the value of k for which f has a critical point at $x = -5$.
- (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f .
Find $\int f(x) dx$.
-

5.5 The Candidates Test for Absolute Extrema

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS absolute (global) extrema 绝对极值 • critical point 临界点

Objective

Build the skills to answer exam questions on **the Candidates Test for absolute extrema**.

You must be able to:

- find the **absolute (global) extrema** 绝对极值 on a closed interval

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ The Candidates Test

On a closed interval $[a, b]$, evaluate f at **all critical points and both endpoints**. The largest value is the absolute maximum; the smallest is the absolute minimum.

■ Example

$f(x) = x^2$ on $[-1, 3]$: critical point $x = 0$; candidates $f(-1) = 1$, $f(0) = 0$, $f(3) = 9$; absolute max 9, absolute min 0.

2 Practice

2.1 State what to evaluate for absolute extrema on $[a, b]$. [1]

2.2 For $f(x) = x^2$ on $[-1, 3]$, list the candidate inputs. [2]

2.3 State how to pick the absolute maximum. [1]

3 Exam-style questions

3.1 To find absolute extrema on $[a, b]$, evaluate f at [1]

- **A** only the endpoints
 - **B** the critical points and endpoints
 - **C** only the critical points
 - **D** $x = 0$
-

3.2 The absolute maximum is the [1]

- **A** smallest candidate value
 - **B** largest candidate value
 - **C** midpoint
 - **D** average
-

3.3 $f(x) = x^2$ on $[-1, 3]$.

(a) State the critical point. [1]

(b) Evaluate at the candidates $-1, 0, 3$. [1]

(c) State the absolute maximum. [1]

Solutions

2.1 f at all critical points and the two endpoints.

2.2 $x = -1$, $x = 0$, $x = 3$.

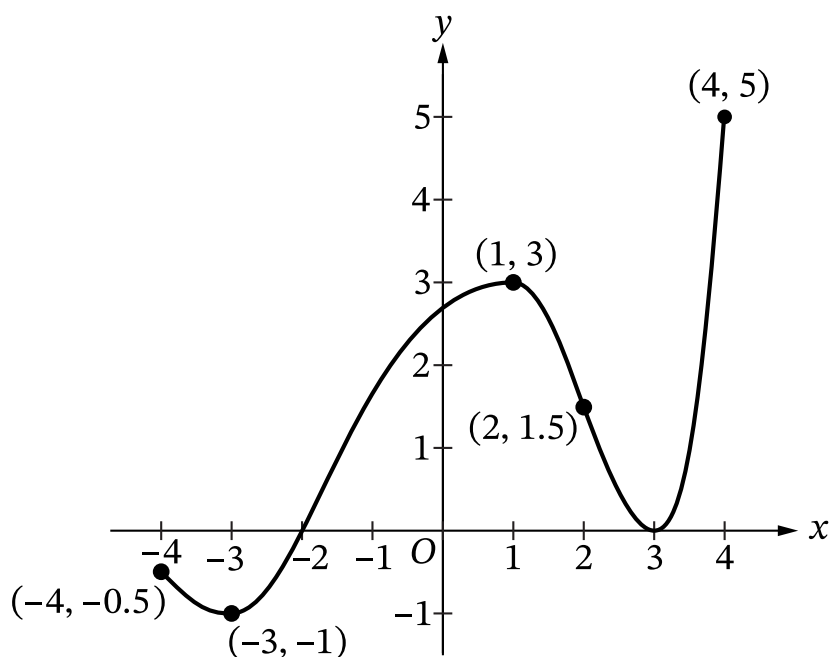
2.3 the largest of the candidate values.

3.1 B.

3.2 B.

3.3 (a) $x = 0$. (b) $f(-1) = 1$, $f(0) = 0$, $f(3) = 9$. (c) 9.

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f' , the derivative of f , is shown.

Graph of f' **Part A**

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Part C

For $-4 \leq x \leq 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

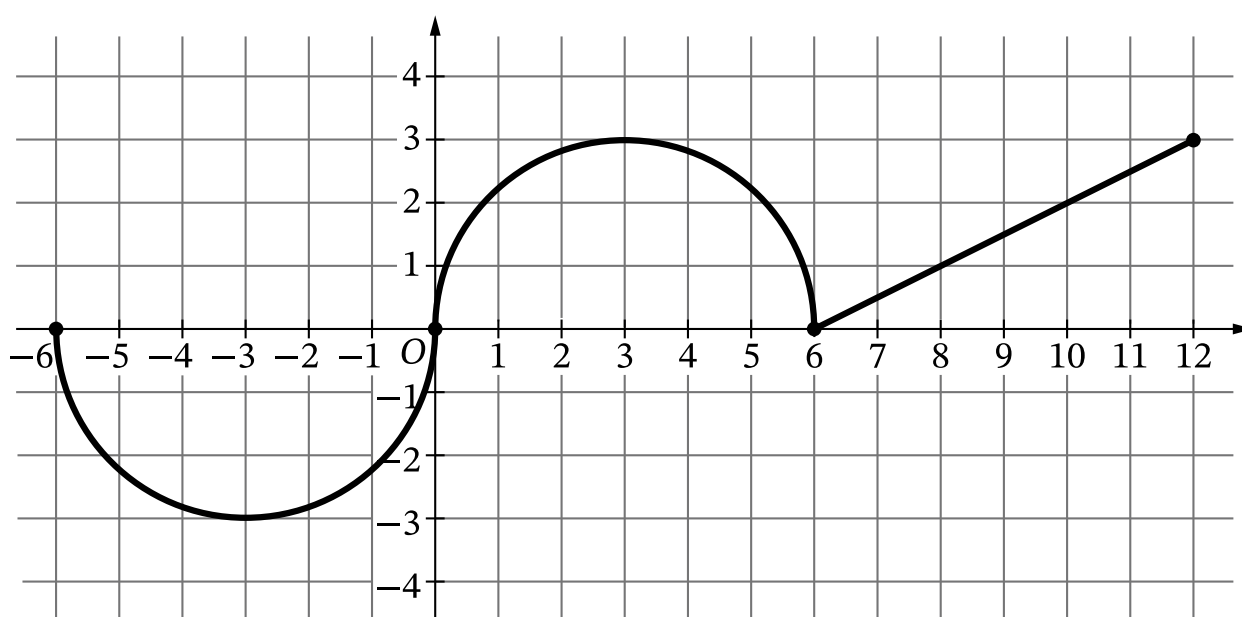
Part D

For $-4 \leq x \leq 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6 \arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

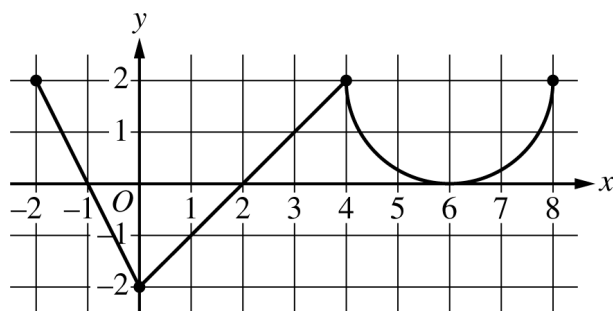
- A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.
- B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \leq t \leq 4$. Show the setup for your calculations.
- C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.
- D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A , defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x) dx$, models the number of acres affected by the species over the time interval $4 \leq t \leq 36$. At what time t , for $4 \leq t \leq 36$, does A attain its maximum value? Justify your answer.
4. The continuous function f is defined on the closed interval $-6 \leq x \leq 12$. The graph of f , consisting of two semicircles and one line segment, is shown in the figure.



Graph of f

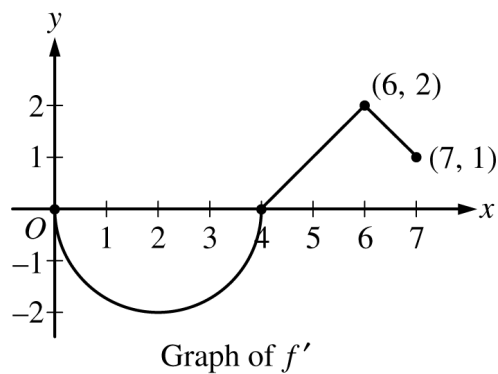
Let g be the function defined by $g(x) = \int_6^x f(t) dt$.

- A. Find $g'(8)$. Give a reason for your answer.
- B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.
- C. Find $g(12)$ and $g(0)$. Label your answers.
- D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \leq x \leq 12$. Justify your answer.

Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f' , the derivative of f , consists of two line segments and a semicircle, as shown in the figure.
- (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.
- (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.
- (c) Find the value of $\lim_{x \rightarrow 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.
- (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f' , the derivative of f , consists of a semicircle and two line segments, as shown in the figure above.
- (a) Find $f(0)$ and $f(5)$.
- (b) Find the x -coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.
- (c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
- (d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

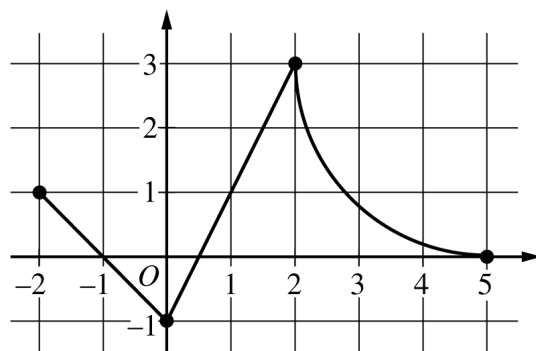
1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15 \sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).
- (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.
- (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?
- (c) At what time t , for $0 \leq t \leq 8$, is the greatest number of fish in the lake? Justify your answer.
- (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.
-

CALCULUS BC
SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f , consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f .
- (a) If $\int_{-6}^5 f(x) \, dx = 7$, find the value of $\int_{-6}^{-2} f(x) \, dx$. Show the work that leads to your answer.
- (b) Evaluate $\int_3^5 (2f'(x) + 4) \, dx$.
- (c) The function g is given by $g(x) = \int_{-2}^x f(t) \, dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.
- (d) Find $\lim_{x \rightarrow 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.
-

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \leq t \leq 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

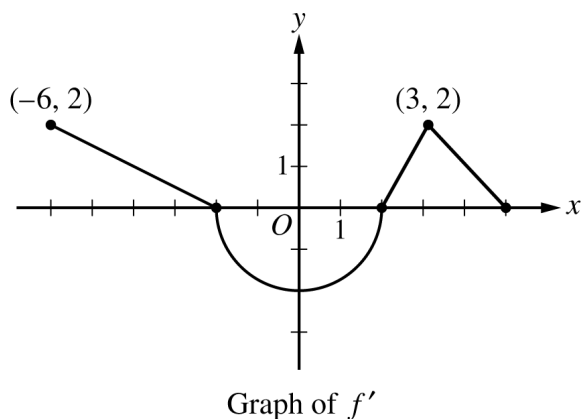
- (a) How many people enter the line for the escalator during the time interval $0 \leq t \leq 300$?
- (b) During the time interval $0 \leq t \leq 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?
- (c) For $t > 300$, what is the first time t that there are no people in line for the escalator?
- (d) For $0 \leq t \leq 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.
-

CALCULUS BC
SECTION II, Part B

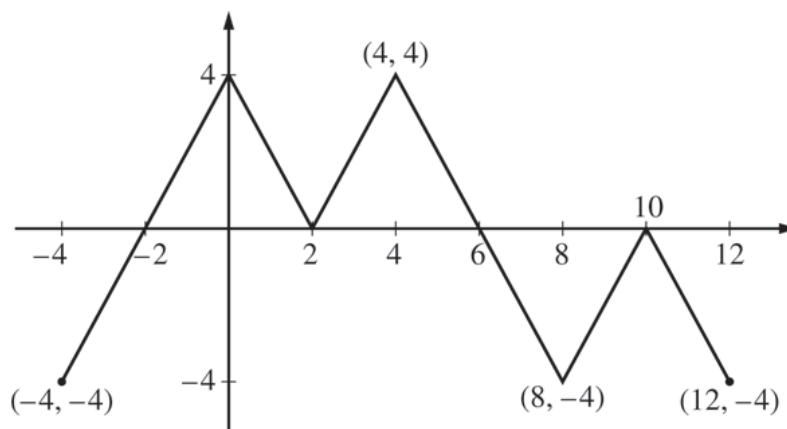
Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f' , the derivative of f , consists of a semicircle and three line segments, as shown in the figure above.
- (a) Find the values of $f(-6)$ and $f(5)$.
- (b) On what intervals is f increasing? Justify your answer.
- (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.
- (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.
-

CALCULUS BC
SECTION II, Part B**Time—60 minutes****Number of problems—4****No calculator is allowed for these problems.**Graph of f

3. The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t) dt$.
- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.
-

CALCULUS BC
SECTION II, Part A**Time—30 minutes****Number of problems—2****A graphing calculator is required for these problems.**

1. The rate at which rainwater flows into a drainpipe is modeled by the function R , where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.
- (a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?
- (b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.
- (c) At what time t , $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.
- (d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.
-

5.6 Concavity and Points of Inflection

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS concave down 下凹 • concave up 上凹 • concavity 凹凸性 • inflection point 拐点

Objective

Build the skills to answer exam questions on **determining concavity**.

You must be able to:

- use the sign of f'' to find **concavity** 凹凸性
- identify an **inflection point** 拐点

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Concavity

- f is **concave up** where $f'' > 0$;
- f is **concave down** where $f'' < 0$.

An **inflection point** is where concavity changes (and f'' changes sign).

■ Example

$f(x) = x^3$: $f''(x) = 6x$, which is > 0 for $x > 0$; the inflection point is at $x = 0$.

2 Practice

2.1 State where f is concave up. [1]

2.2 For $f(x) = x^3$, find f'' and where it is concave up. [2]

2.3 State what an inflection point is. [1]

3 Exam-style questions

3.1 f is concave up where [1]

- **A** $f' > 0$
 - **B** $f'' > 0$
 - **C** $f'' < 0$
 - **D** $f = 0$
-

3.2 An inflection point is where [1]

- **A** $f' = 0$
 - **B** concavity changes
 - **C** $f = 0$
 - **D** f'' is constant
-

3.3 $f(x) = x^3$.

(a) Find $f''(x)$. [1]

(b) State where $f'' > 0$. [1]

(c) State the inflection point. [1]

Solutions

2.1 where $f'' > 0$.

2.2 $f''(x) = 6x$; concave up for $x > 0$.

2.3 a point where concavity changes (from up to down or vice versa).

3.1 B.

3.2 B.

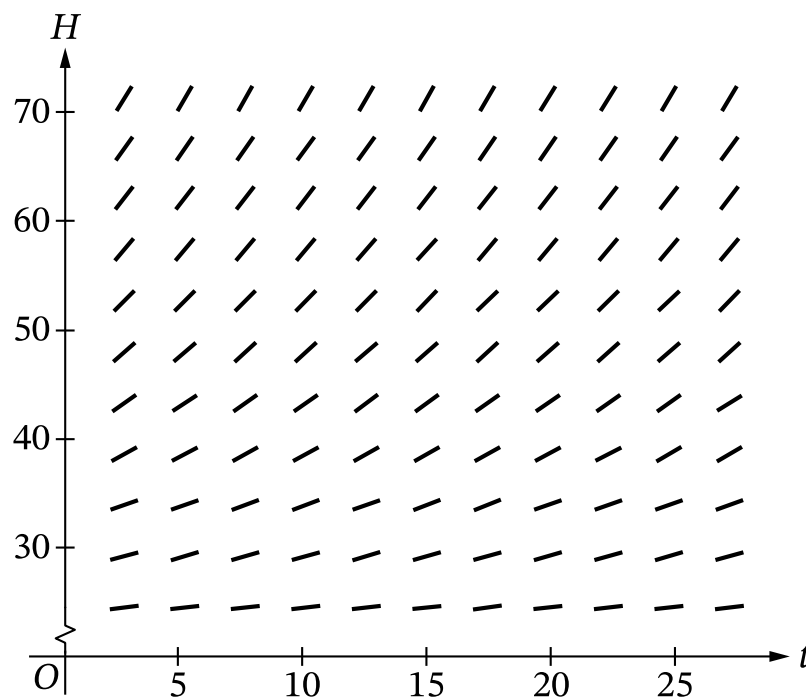
3.3 (a) $6x$. (b) $x > 0$. (c) $x = 0$.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation

$$\frac{dH}{dt} = -\frac{1}{15}(H - 20).$$

**Part B**

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

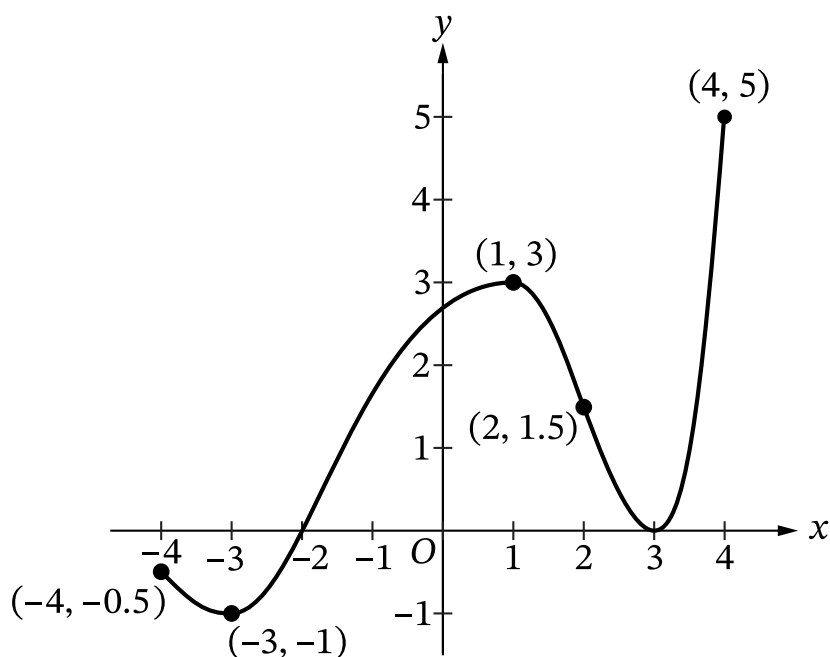
Part C

It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f' , the derivative of f , is shown.

Graph of f' **Part A**

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

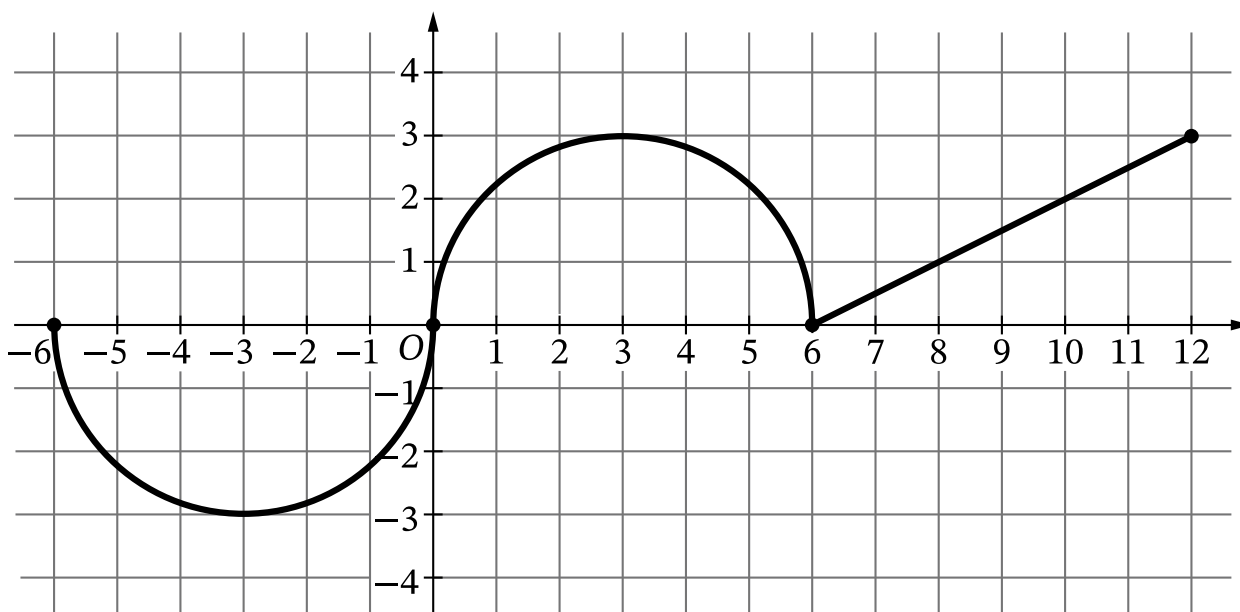
Part C

For $-4 \leq x \leq 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

Part D

For $-4 \leq x \leq 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

4. The continuous function f is defined on the closed interval $-6 \leq x \leq 12$. The graph of f , consisting of two semicircles and one line segment, is shown in the figure.

Graph of f

Let g be the function defined by $g(x) = \int_6^x f(t) dt$.

- A. Find $g'(8)$. Give a reason for your answer.
- B. Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.
- C. Find $g(12)$ and $g(0)$. Label your answers.
- D. Find the value of x at which g attains an absolute minimum on the closed interval $-6 \leq x \leq 12$. Justify your answer.

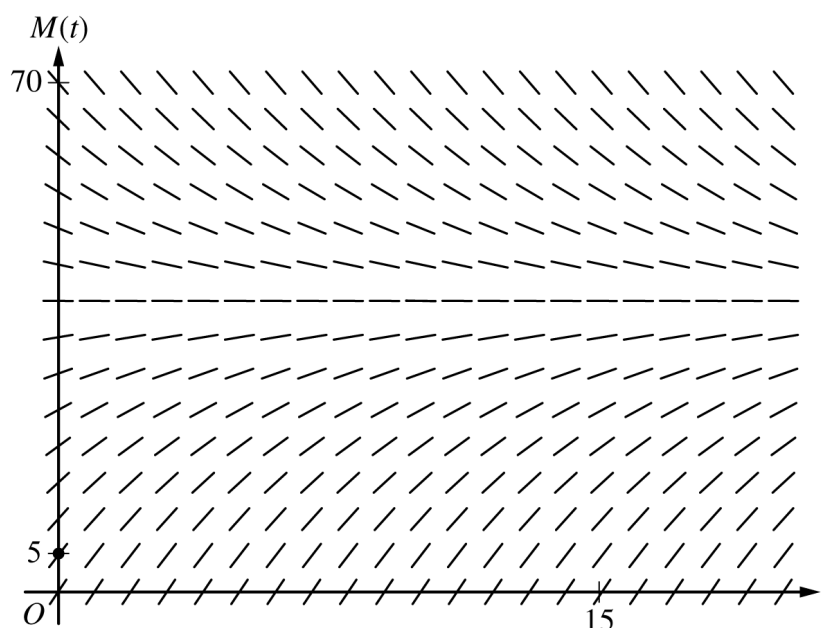
t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C , where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.
- (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.
- (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t) \, dt$. Interpret the meaning of $\frac{1}{12} \int_0^{12} C(t) \, dt$ in the context of the problem.
- (c) For $12 \leq t \leq 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$.
Show the setup for your calculations.
- (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate.
Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

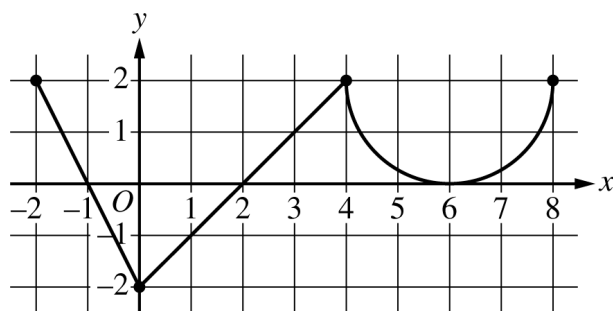
3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t , where $M(t)$ is measured in degrees Celsius ($^{\circ}\text{C}$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C . It can be shown that $M(t) < 40$ for all values of t .

- (a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



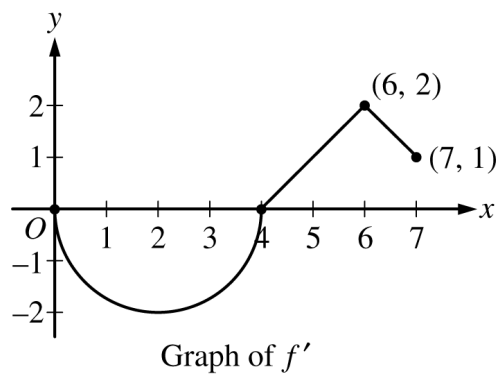
- (b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.
- (c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M . Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.
- (d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Graph of f'

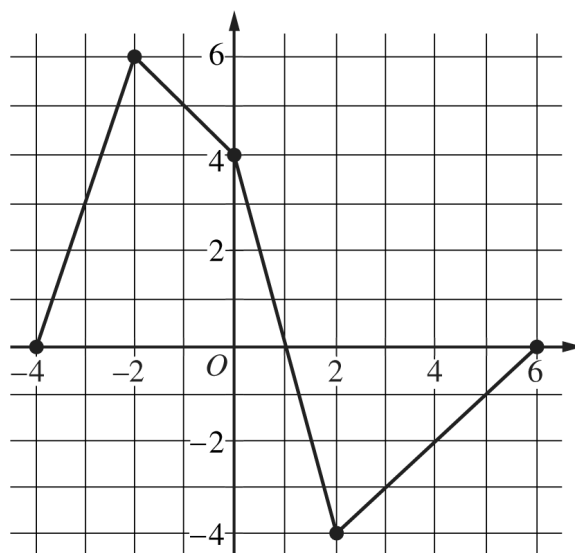
4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f' , the derivative of f , consists of two line segments and a semicircle, as shown in the figure.
- (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.
- (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.
- (c) Find the value of $\lim_{x \rightarrow 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.
- (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f' , the derivative of f , consists of a semicircle and two line segments, as shown in the figure above.
- (a) Find $f(0)$ and $f(5)$.
- (b) Find the x -coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.
- (c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
- (d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

Graph of f

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f , consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t) \, dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \rightarrow 2} \frac{G(x)}{x^2 - 2x}$.

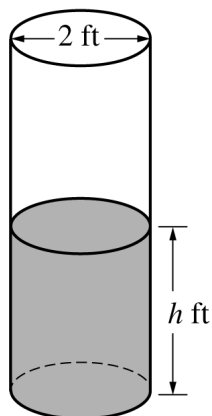
(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c , $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15 \sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).
- (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.
- (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?
- (c) At what time t , for $0 \leq t \leq 8$, is the greatest number of fish in the lake? Justify your answer.
- (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.
-



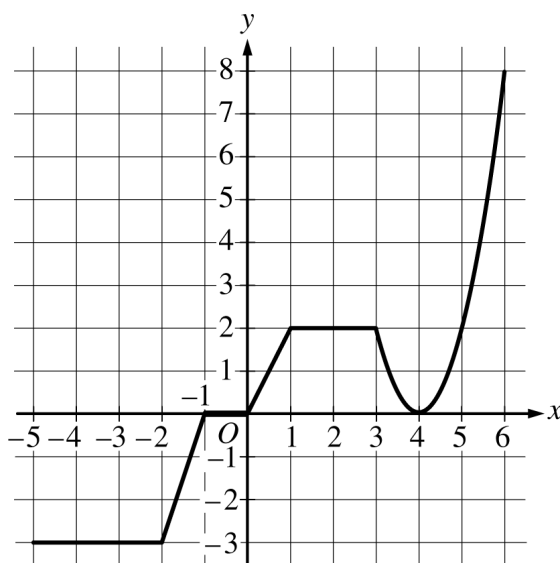
4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
- (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.
- (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
- (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .
-

CALCULUS BC
SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The graph of the continuous function g , the derivative of the function f , is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \leq x \leq 6$.
- (a) If $f(1) = 3$, what is the value of $f(-5)$?
- (b) Evaluate $\int_1^6 g(x) \, dx$.
- (c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.
- (d) Find the x -coordinate of each point of inflection of the graph of f . Give a reason for your answer.
-

CALCULUS BC
SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A , where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.
- (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.
- (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.
- (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \frac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.
- (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.
-

5.7 The Second Derivative Test for Extrema

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS local extremum 局部极值 • critical point 临界点 • local minimum 极小值 • local maximum 极大值
• concave down 下凹

Objective

Build the skills to answer exam questions on **the Second Derivative Test**.

You must be able to:

- use f'' at a critical point to classify a **local extremum** 局部极值

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Second Derivative Test

At a critical point c where $f'(c) = 0$:

- $f''(c) > 0 \rightarrow$ local **minimum** (concave up);
- $f''(c) < 0 \rightarrow$ local **maximum** (concave down);
- $f''(c) = 0 \rightarrow$ **inconclusive**.

■ Example

$f(x) = x^2 - 4x$: $f''(x) = 2 > 0$, so the critical point is a local minimum.

2 Practice

2.1 State the Second Derivative Test for a local minimum. [1]

2.2 For $f(x) = x^2 - 4x$ with $f'' = 2$, classify the critical point. [2]

2.3 State when the test is inconclusive. [1]

3 Exam-style questions

3.1 If $f'(c) = 0$ and $f''(c) < 0$, then c is a [1]

- **A** local minimum
 - **B** local maximum
 - **C** inflection point
 - **D** nothing
-

3.2 If $f'(c) = 0$ and $f''(c) > 0$, then c is a [1]

- **A** local maximum
 - **B** local minimum
 - **C** endpoint
 - **D** asymptote
-

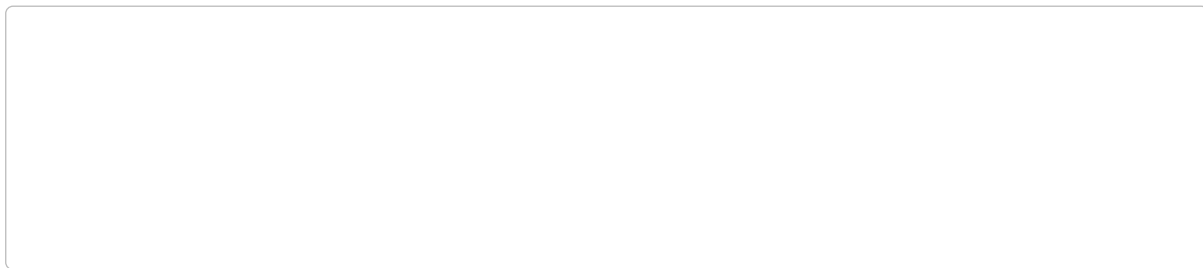
3.3 $f(x) = x^2 - 6x$, with $f'(3) = 0$.

(a) Find $f''(x)$. [1]

(b) Evaluate $f''(3)$. [1]

(c) Classify $x = 3$.

[1]



Solutions

2.1 if $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum.

2.2 $f'' = 2 > 0$, so it is a local minimum.

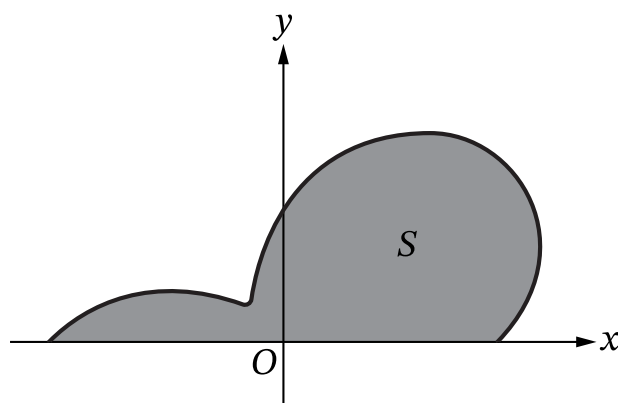
2.3 when $f''(c) = 0$.

3.1 B.

3.2 B.

3.3 (a) 2. (b) $2 > 0$. (c) local minimum.

2. Let S be the shaded region bounded by the graph of the polar curve $r(\theta) = 3 + 2 \sin(2\theta) + \cos(2\theta)$ for $0 \leq \theta \leq \pi$, as shown in the figure.



It can be shown that $r'(\theta) = 4 \cos(2\theta) - 2 \sin(2\theta)$.

(Note: Your calculator should be in radian mode.)

Part A

Find the area of region S . Show the setup for your calculations.

Part B

There is a point on the curve at which the slope of the line tangent to the curve is $-\frac{3}{7}$. At this point, $\frac{dy}{d\theta} = \frac{3\sqrt{2}}{2}$. Find $\frac{dx}{d\theta}$ at this point. Show the work that leads to your answer.

Part C

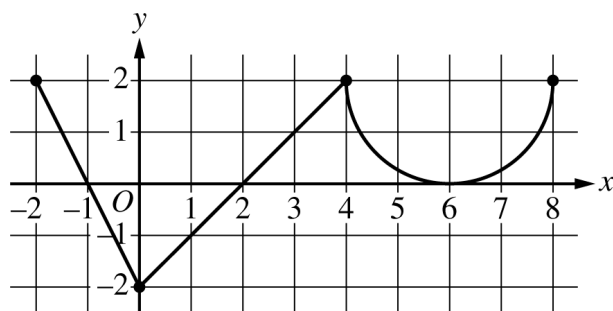
- Find the value of θ in the interval $0 < \theta < \frac{\pi}{2}$ at which r has a critical point.
- Use a derivative test to determine whether the critical point is the location of a relative minimum, a relative maximum, or neither for r .

Part D

Find the average distance from the origin to a point on the polar curve

$r(\theta) = 3 + 2 \sin(2\theta) + \cos(2\theta)$ for $\frac{\pi}{2} \leq \theta \leq \pi$. Show the setup for your calculations.

END OF PART A

Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f' , the derivative of f , consists of two line segments and a semicircle, as shown in the figure.
- (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.
- (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.
- (c) Find the value of $\lim_{x \rightarrow 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.
- (d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.
- (a) Find $\frac{d^2y}{dx^2}$ in terms of x and y .
- (b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.
- (c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.
- (d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

5.8 Sketching a Function and Its Derivative

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS increasing 递增 • concave up 上凹 • local minimum 极小值 • decreasing 递减

Objective

Build the skills to answer exam questions on **sketching functions and their derivatives**.

You must be able to:

- relate features of f to the sign and zeros of f'

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Sketching f and f'

- where f has a local max/min, $f' = 0$ (crosses the axis);
- where f is increasing, $f' > 0$ (above the axis);
- where f is concave up, f' is increasing.

Use these links to sketch one graph from the other.

2 Practice

2.1 State the value of f' at a local extremum of f . [1]

2.2 State the sign of f' where f is increasing. [1]

2.3 State what f' does where f is concave up. [2]

3 Exam-style questions

3.1 At a local max of f , f' is [1]

- **A** positive
 - **B** zero
 - **C** negative always
 - **D** undefined
-

3.2 Where f is decreasing, its derivative f' is [1]

- **A** positive
 - **B** negative
 - **C** zero
 - **D** increasing
-

3.3 f has a local minimum at $x = 2$.

(a) State $f'(2)$. [1]

(b) State the sign of f' just left of 2. [1]

(c) State the sign just right of 2. [1]

Solutions

2.1 $f' = 0$.

2.2 positive ($f' > 0$).

2.3 f' is increasing.

3.1 B.

3.2 B.

3.3 (a) 0. (b) negative. (c) positive.

5.9 Connecting f , f' , and f''

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS concave up 上凹 • increasing 递增 • local minimum 极小值 • decreasing 递减

Objective

Build the skills to answer exam questions on **connecting a function, its first derivative, and its second derivative**.

You must be able to:

- read f , f' , and f'' together to describe a graph

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Connecting f , f' , f''

- f' tells increasing/decreasing;
- f'' tells concavity;
- a local **min** of $f \Leftrightarrow f' = 0$ and $f'' > 0$;
- an **inflection** point of $f \Leftrightarrow f''$ changes sign.

■ Example

$f(x) = x^3$: $f'(x) = 3x^2$ (zero at 0), $f''(x) = 6x$ (zero at 0).

2 Practice

2.1 State what f'' tells you about f . [1]

2.2 State the condition on f' and f'' at a local minimum. [1]

2.3 For $f(x) = x^3$, state where $f' = 0$ and where $f'' = 0$. [2]

3 Exam-style questions

3.1 f'' gives information about a graph's [1]

- **A** slope
 - **B** concavity
 - **C** x -intercepts
 - **D** domain
-

3.2 A local minimum of f satisfies [1]

- **A** $f' = 0$, $f'' > 0$
 - **B** $f' = 0$, $f'' < 0$
 - **C** $f'' = 0$ only
 - **D** $f = 0$
-

3.3 $f(x) = x^3$.

(a) Find $f'(x)$. [1]

(b) State where $f' = 0$. [1]

(c) State where $f'' = 0$. [1]

Solutions

2.1 its concavity (concave up where $f'' > 0$, down where $f'' < 0$).

2.2 $f' = 0$ and $f'' > 0$.

2.3 $f' = 0$ at $x = 0$; $f'' = 0$ at $x = 0$.

3.1 B.

3.2 A.

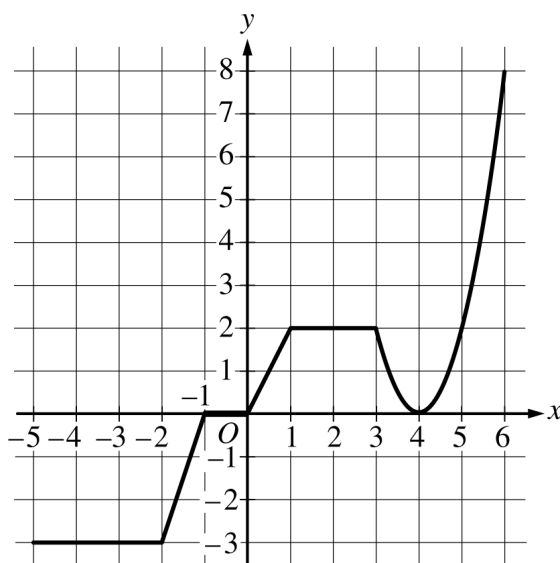
3.3 (a) $3x^2$. (b) $x = 0$. (c) $x = 0$.

CALCULUS BC
SECTION II, Part B

Time—1 hour

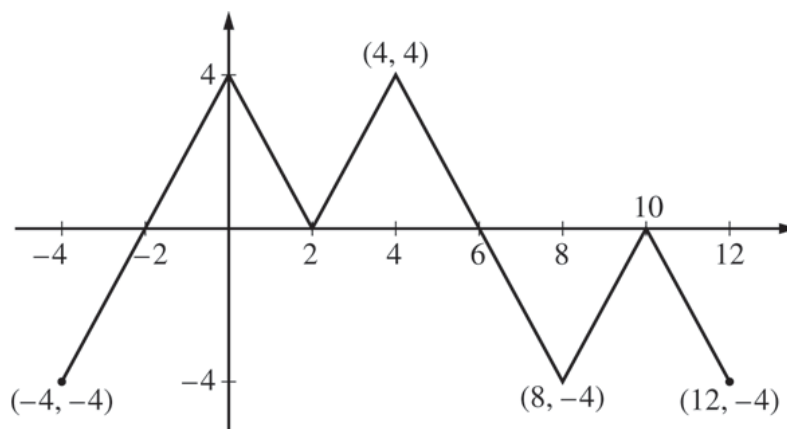
Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

3. The graph of the continuous function g , the derivative of the function f , is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x - 4)^2$ for $3 \leq x \leq 6$.
- (a) If $f(1) = 3$, what is the value of $f(-5)$?
- (b) Evaluate $\int_1^6 g(x) \, dx$.
- (c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.
- (d) Find the x -coordinate of each point of inflection of the graph of f . Give a reason for your answer.
-

CALCULUS BC
SECTION II, Part B**Time—60 minutes****Number of problems—4****No calculator is allowed for these problems.**Graph of f

3. The figure above shows the graph of the piecewise-linear function f . For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t) dt$.
- (a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
- (b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
- (c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.
- (d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.
-

5.10 Introduction to Optimization Problems

Name: _____ Class: _____ Date: _____

Total: 12 marks

KEY WORDS constraint 约束 • optimization 最优化 • objective function 目标函数

Objective

Build the skills to answer exam questions on **optimization** 最优化 — turning a word problem into a function and finding its greatest or least value.

You must be able to:

- write a quantity to be maximized or minimized as a function of one variable (the **objective function** 目标函数)
- use a **constraint** 约束 to eliminate the extra variable
- state the **domain** that makes physical sense for the problem

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Naming the objective and the constraint

Optimization always has two ingredients: the **objective** (what you want biggest or smallest) and a **constraint** (a fixed condition). A farmer has 40 m of fence for a rectangular pen against a wall (no fence on the wall side).

- Objective: **area** $A = xy$ (maximize).
- Constraint: fence used $= x + 2y = 40$.

■ Reducing to one variable

Solve the constraint for one variable and substitute. From $x + 2y = 40$, $x = 40 - 2y$, so

$$A(y) = (40 - 2y)y = 40y - 2y^2.$$

Now A depends on the single variable y — ready to differentiate.

■ **Stating a sensible domain**

Lengths are positive, so $y > 0$ and $x = 40 - 2y > 0$, giving $0 < y < 20$. The optimization only searches this **domain**; an answer outside it is rejected.

■ **Reading what the question asks for**

Check whether the question wants the **variable** (the y that optimizes), the **maximum value** (A itself), or both. A common lost mark is giving y when the question asked for the area.

2 Practice

Now apply the methods above.

2.1 A rectangle has perimeter 24 cm. Write its **area** as a function of its width x . [2]

2.2 An open-top box has a square base of side x and volume 32 cm^3 . Write its **surface area** (base + 4 sides) as a function of x . [2]

2.3 For the farmer's pen $A(y) = 40y - 2y^2$, state the domain of y . [1]

3 Exam-style questions

3.1 A quantity Q is to be maximized subject to a constraint. What is the **first** step? [1]

- **A** differentiate Q immediately
- **B** write Q as a function of a single variable using the constraint
- **C** set $Q = 0$
- **D** find the second derivative

3.2 A cylindrical can has volume 355 cm^3 . Its surface area is $S = 2\pi r^2 + 2\pi rh$.

(a) Use the volume constraint $\pi r^2 h = 355$ to write S as a function of r only. [2]

(b) State the domain of r .

[1]

3.3 A 600 cm^2 sheet of card is folded into an open box with a square base of side x and height h , using all the card ($x^2 + 4xh = 600$). Write the **volume** V as a function of x alone. [3]

Solutions

2.1 Perimeter $2x + 2w = 24 \Rightarrow \text{length} = 12 - x$; $A = x(12 - x) = 12x - x^2$.

2.2 Height from $x^2h = 32$, so $h = 32/x^2$; $S = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$.

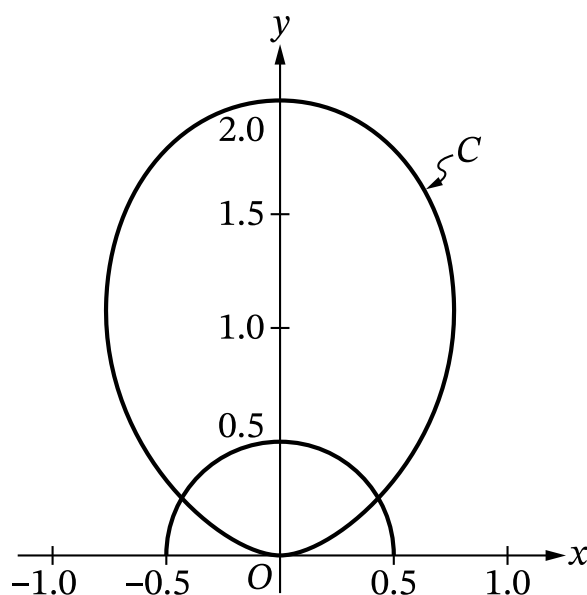
2.3 $0 < y < 20$.

3.1 B — reduce to a single variable using the constraint before differentiating.

3.2 (a) $h = \frac{355}{\pi r^2}$; $S(r) = 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right) = 2\pi r^2 + \frac{710}{r}$. (b) $r > 0$.

3.3 $h = \frac{600 - x^2}{4x}$; $V = x^2h = x^2\left(\frac{600 - x^2}{4x}\right) = \frac{600x - x^3}{4} = 150x - \frac{1}{4}x^3$.

2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \leq \theta \leq \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \leq \theta \leq \pi$ are shown in the xy -plane.



(Note: Your calculator should be in radian mode.)

- A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.
- B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.
- C. It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C . For $0 \leq \theta \leq \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y -axis. Justify your answer.
- D. A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t . Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

END OF PART A

5.11 Solving Optimization Problems

Name: _____ Class: _____ Date: _____

Total: 17 marks

KEY WORDS critical points 临界点 • objective function 目标函数 • local maximum 极大值 • optimization 最优化
• point of inflection 拐点

Objective

Build the skills to answer exam questions on **solving optimization problems** — carrying the setup through to a justified maximum or minimum.

You must be able to:

- differentiate the **objective function** 目标函数 and solve $f'(x) = 0$ for the **critical points** 临界点
- justify** that a critical point is a maximum or minimum (sign of f' , or the second-derivative test, or the closed-interval candidates)
- give the answer the question asks for, with units

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The full optimization method

Maximize the pen area $A(y) = 40y - 2y^2$ on $0 < y < 20$.

- Differentiate: $A'(y) = 40 - 4y$.
- Solve $A'(y) = 0$: $y = 10$.
- Justify: $A''(y) = -4 < 0$, so $y = 10$ gives a **maximum**.
- Answer: $x = 40 - 2(10) = 20$, so $A = 20 \times 10 = 200 \text{ m}^2$.

■ Justifying with the first-derivative sign

If the second derivative is awkward, test the sign of f' on each side of the critical point. Here $A'(9) = +4 > 0$ and $A'(11) = -4 < 0$: f rises then falls, so the critical point is a **maximum**. A justification is **required** for full marks.

■ Minimizing a cost

A can's surface area is $S(r) = 2\pi r^2 + \frac{710}{r}$. Then $S'(r) = 4\pi r - \frac{710}{r^2}$. Setting $S' = 0$: $4\pi r^3 = 710$, so $r = \left(\frac{710}{4\pi}\right)^{1/3} \approx 3.84$ cm. Since $S''(r) = 4\pi + \frac{1420}{r^3} > 0$, this is a **minimum**.

2 Practice

Now apply the methods above.

2.1 For $A(y) = 40y - 2y^2$, find $A'(y)$ and solve $A'(y) = 0$. [2]

2.2 The area function is $A = 12x - x^2$. Find the width x that maximizes the area, and **justify** it is a maximum. [3]

2.3 A function $C(x) = x^2 + \frac{16}{x}$ models a cost for $x > 0$. Find the x that minimizes C . [3]

3 Exam-style questions

3.1 A critical point of f has $f'(c) = 0$ and $f''(c) < 0$. The point is a [1]

- **A** local minimum
 - **B** local maximum
 - **C** point of inflection
 - **D** endpoint
-

3.2 A rectangular field of area $A = x(200 - x)$ m² is enclosed, where $0 < x < 200$.

(a) Find the value of x that maximizes A . [2]

(b) Justify that your value gives a maximum. [1]

(c) State the maximum area. [1]

3.3 An open box is made from card, with volume $V(x) = 150x - \frac{1}{4}x^3$ for $0 < x < \sqrt{600}$.

(a) Show that $V'(x) = 150 - \frac{3}{4}x^2$. [1]

(b) Find the value of x that maximizes the volume, and justify it. [3]

Solutions

2.1 $A'(y) = 40 - 4y$; $40 - 4y = 0 \Rightarrow y = 10$.

2.2 $A'(x) = 12 - 2x = 0 \Rightarrow x = 6$; $A''(x) = -2 < 0$, so $x = 6$ is a maximum.

2.3 $C'(x) = 2x - \frac{16}{x^2} = 0$; $2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

3.1 B — $f'(c) = 0$ with $f''(c) < 0$ is a local maximum.

3.2 (a) $A = 200x - x^2$, $A'(x) = 200 - 2x = 0 \Rightarrow x = 100$. (b) $A''(x) = -2 < 0$, so it is a maximum. (c) $A = 100(100) = 10\,000 \text{ m}^2$.

3.3 (a) $V'(x) = 150 - \frac{3}{4}x^2$. (b) $150 - \frac{3}{4}x^2 = 0 \Rightarrow x^2 = 200 \Rightarrow x = \sqrt{200} \approx 14.1$; $V''(x) = -\frac{3}{2}x < 0$ for $x > 0$, so it is a maximum.

5.12 Exploring Behaviors of Implicit Relations

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS horizontal tangent 水平切线 • vertical tangent 垂直切线 • concavity 凹凸性 • concave down 下凹
• concave up 上凹

Objective

Build the skills to answer exam questions on the **behavior of implicit relations** — using $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ found by implicit differentiation to locate features of a curve.

You must be able to:

- find where an implicit curve has a **horizontal tangent** 水平切线 ($\frac{dy}{dx} = 0$) or a **vertical tangent** 垂直切线 ($\frac{dy}{dx}$ undefined)
- decide **concavity** 凹凸性 from the sign of $\frac{d^2y}{dx^2}$ on the curve
- evaluate these using a point that lies **on** the curve

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Horizontal tangents

For $x^2 + y^2 = 25$, implicit differentiation gives $2x + 2y \frac{dy}{dx} = 0$, so

$$\frac{dy}{dx} = -\frac{x}{y}.$$

A **horizontal** tangent needs $\frac{dy}{dx} = 0$, i.e. the numerator $x = 0$ (with $y \neq 0$): the points $(0, 5)$ and $(0, -5)$.

■ Vertical tangents

A **vertical** tangent occurs where $\frac{dy}{dx}$ is undefined — the denominator zero, $y = 0$ (with $x \neq 0$): the points $(5, 0)$ and $(-5, 0)$. This matches the circle's shape.

■ Concavity on an implicit curve

Differentiate $\frac{dy}{dx} = -\frac{x}{y}$ again (quotient + chain rule):

$$\frac{d^2y}{dx^2} = -\frac{y - x \frac{dy}{dx}}{y^2} = -\frac{y - x(-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{25}{y^3}.$$

Where $y > 0$ this is negative (**concave down**, the top of the circle); where $y < 0$ it is positive (**concave up**).

2 Practice

Now apply the methods above.

2.1 For $x^2 + y^2 = 25$, state $\frac{dy}{dx}$. [1]

2.2 Find the point(s) on $x^2 + y^2 = 25$ where the tangent is **horizontal**. [2]

2.3 A curve satisfies $\frac{dy}{dx} = \frac{2x}{y}$. State where its tangent line is **vertical**. [1]

3 Exam-style questions

3.1 A curve has $\frac{dy}{dx} = \frac{x-1}{y}$. A horizontal tangent occurs where [1]

- **A** $y = 0$
 - **B** $x = 0$
 - **C** $x = 1$
 - **D** $x = y$
-

3.2 The curve $x^2 + xy + y^2 = 7$ passes through $(1, 2)$. Its derivative is $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$.

(a) Find the slope of the tangent at $(1, 2)$. [2]

(b) Determine whether the tangent at $(1, 2)$ is horizontal, vertical, or neither. [1]

3.3 For $x^2 + y^2 = 25$ it can be shown that $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$. State, with a reason, whether the curve is concave up or concave down at the point $(3, 4)$. [2]

Solutions

2.1 $\frac{dy}{dx} = -\frac{x}{y}$.

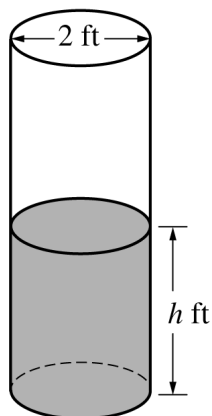
2.2 Set numerator $x = 0$; points $(0, 5)$ and $(0, -5)$.

2.3 Vertical where denominator $y = 0$, i.e. at $(\pm 5, 0)$ — where the curve meets the x -axis.

3.1 C — a horizontal tangent needs the numerator zero, $x - 1 = 0$, so $x = 1$.

3.2 (a) $\frac{dy}{dx} = -\frac{2(1) + 2}{1 + 2(2)} = -\frac{4}{5}$. (b) Neither — the slope $-\frac{4}{5}$ is finite and non-zero.

3.3 $\frac{d^2y}{dx^2} = -\frac{25}{4^3} = -\frac{25}{64} < 0$; negative, so the curve is **concave down** at $(3, 4)$.



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
- (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.
- (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
- (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .
-