Week 10

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

AP Calculus BC Name: ______________________ Score: ______

1. $V(t)$ is water volume in litres at time t minutes. The units of $V'(t)$ are...
   - ☐ litres
   - ☐ litres per minute
   - ☐ minutes
   - ☐ minutes per litre
2. Velocity is the derivative of...
   - ☐ acceleration
   - ☐ position
   - ☐ speed
   - ☐ time
3. $C(x)$ is cost in dollars for x items. The units of the marginal cost $C'(x)$ are...
   - ☐ dollars
   - ☐ dollars per item
   - ☐ items
   - ☐ items per dollar
4. A related-rates problem links the rates of quantities that change with respect to...
   - ☐ each other only
   - ☐ time
   - ☐ temperature
   - ☐ nothing
5. With $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, $r = 5$, and $\frac{dr}{dt} = 2$, find $\frac{dV}{dt}$ as a multiple of π (enter the number before π).

   ____
6. The linearization of f at a is $L(x) =$
   - ☐ $f(a) + f'(a)(x - a)$
   - ☐ $f'(a) + f(a)(x - a)$
   - ☐ $f(a) \cdot f'(a)$
   - ☐ $f(x) - f(a)$

7. L'Hospital's Rule applies to **which** indeterminate forms?

   - ☐ $\frac{0}{0}$
   - ☐ $\frac{\infty}{\infty}$
   - ☐ $\frac{5}{2}$
   - ☐ $\frac{3}{0}$

   Tick every correct option.

8. For $v(t) = 3t^2 - 12t + 9$, the object is at rest when $v = 0$. One such time is $t = 1$; find the **other** time.

   ____

9. For $V(t) = 100 - 4t^2$, the rate is $V'(t) = -8t$. Find $V'(3)$ (litres per minute).

   ____

10. Using $L(x) = 2 + \frac{1}{4}(x - 4)$, estimate $\sqrt{4.1} = L(4.1)$.

   ____

AP Calculus BC

1. **litres per minute**

   Output per input: litres per minute.

2. **position**

   $v = s'$: velocity is the derivative of position.

3. **dollars per item**

   Output per input: dollars per item.

4. **time**

   The variables are functions of time, linked via the chain rule.

5. **200**

   $4\pi(25)(2) = 200\pi$.

6. $f(a) + f'(a)(x - a)$

   Point-slope of the tangent: $f(a) + f'(a)(x - a)$.

7. $\frac{0}{0}$**;** $\frac{\infty}{\infty}$

   Only $\frac{0}{0}$ and $\frac{\infty}{\infty}$. $\frac{5}{2}$ is already determinate; $\frac{3}{0}$ is an asymptote, not indeterminate.

8. **3**

   $3(t^2 - 4t + 3) = 3(t-1)(t-3) = 0 \Rightarrow t = 1$ or $t = 3$.

9. **-24**

   $V'(3) = -8(3) = -24$; draining at 24 L/min.

10. **2.025**

    $2 + \frac{1}{4}(0.1) = 2.025$.

4.1 Interpreting the Meaning of the Derivative in Context

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS units 单位

Objective

Build the skills to answer exam questions on **interpreting the meaning of the derivative in context**.

You must be able to:

- give the **units 单位** of a derivative
- interpret its sign and value in a real context

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ The derivative in context

A derivative is a **rate of change** with units: if $V(t)$ is a volume in cm^3 at time t (s), then $V'(t)$ is in cm^3/s.

- $f'(t) > 0$: the quantity is increasing;
- $f'(t) < 0$: it is decreasing.

2 Practice

2.1 State the units of $f'(t)$ if f is in metres and t in seconds. [1]

2.2 State what a positive derivative means. [1]

2.3 If $P(t)$ is a population and $P'(5) = 200$, interpret this. [2]

3 Exam-style questions

3.1 If $h(t)$ is a height in m and t in s, then $h'(t)$ has units [1]

- **A** m
- **B** m/s
- **C** s/m
- **D** $\mathrm{m} \cdot \mathrm{s}$

3.2 A positive $f'(t)$ means the quantity is [1]

- **A** decreasing
- **B** increasing
- **C** constant
- **D** zero

3.3 $W(t)$ is water in litres at time t (min), and $W'(3) = -4$.

(a) State the units. [1]

(b) State what the sign means. [1]

(c) Interpret $W'(3) = -4$. [1]

Solutions

2.1 metres per second (m/s).

2.2 the quantity is increasing.

2.3 at $t = 5$, the population is growing at 200 per unit time.

3.1 B.

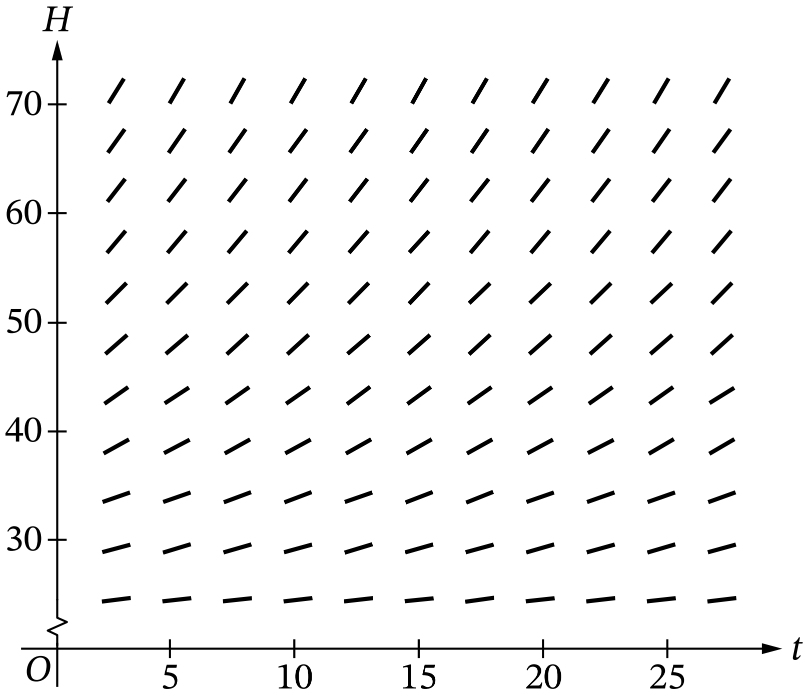
3.2 B.

3.3 (a) litres per minute (L/min). (b) the amount is decreasing. (c) at $t = 3$ min, the water is falling at 4 L/min.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.



Part B

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Part C

It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

2. Researchers on a boat are investigating plankton cells in a sea. At a depth of h meters, the density of plankton cells, in millions of cells per cubic meter, is modeled by $p(h) = 0.2h^2e^{-0.0025h^2}$ for $0 \le h \le 30$ and is modeled by $f(h)$ for $h \ge 30$. The continuous function f is not explicitly given.

(a) Find $p'(25)$. Using correct units, interpret the meaning of $p'(25)$ in the context of the problem.

(b) Consider a vertical column of water in this sea with horizontal cross sections of constant area 3 square meters. To the nearest million, how many plankton cells are in this column of water between $h = 0$ and $h = 30$ meters?

(c) There is a function u such that $0 \le f(h) \le u(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} u(h)\, dh = 105$. The column of water in part (b) is K meters deep, where $K > 30$. Write an expression involving one or more integrals that gives the number of plankton cells, in millions, in the entire column. Explain why the number of plankton cells in the column is less than or equal to 2000 million.

(d) The boat is moving on the surface of the sea. At time $t \ge 0$, the position of the boat is $(x(t), y(t))$, where $x'(t) = 662\sin(5t)$ and $y'(t) = 880\cos(6t)$. Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in meters. Find the total distance traveled by the boat over the time interval $0 \le t \le 1$.

END OF PART A OF SECTION II

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30,$ the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t,$ where $A(t)$ is measured in pounds and t is measured in days.

   (a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30.$ Indicate units of measure.

   (b) Find the value of $A'(15).$ Using correct units, interpret the meaning of the value in the context of the problem.

   (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30.$

   (d) For $t > 30,$ $L(t),$ the linear approximation to A at $t = 30,$ is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

4.2 Straight-Line Motion: Position, Velocity, and Acceleration

Name: ________ Class: ________ Date: ________ **Total: 9 marks**

KEY WORDS velocity 速度 • acceleration 加速度 • at rest 静止 • position 位置 • speed 速率

Objective

Build the skills to answer exam questions on **straight-line motion: position, velocity, and acceleration**.

You must be able to:

- relate **position** 位置, **velocity** 速度, and **acceleration** 加速度 by differentiation
- find when an object is at rest

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Straight-line motion

- velocity $v(t) = x'(t)$;
- acceleration $a(t) = v'(t) = x''(t)$;
- **speed** $= |v(t)|$.

The object is **at rest** when $v = 0$; it moves in the positive direction when $v > 0$.

Example

$x(t) = t^2 - 4t$: $v(t) = 2t - 4 = 0$ at $t = 2$.

2 Practice

2.1 State velocity in terms of position. [1]

2.2 State acceleration in terms of velocity. [1]

2.3 For $x(t) = t^2 - 4t$, find $v(t)$ and the time at rest. [2]

3 Exam-style questions

3.1 Velocity is the derivative of [1]

- **A** acceleration
- **B** position
- **C** speed
- **D** distance

3.2 The object is at rest when [1]

- **A** $a = 0$
- **B** $v = 0$
- **C** $x = 0$
- **D** $t = 0$

3.3 $x(t) = t^2 - 4t$.

(a) Find $v(t)$. [1]

(b) Find $a(t)$. [1]

(c) Find when $v = 0$. [1]

Solutions

2.1 $v(t) = x'(t)$.

2.2 $a(t) = v'(t)$.

2.3 $v(t) = 2t - 4$; at rest when $2t - 4 = 0$, i.e. $t = 2$.

3.1 **B**.

3.2 **B**.

3.3 (a) $v(t) = 2t - 4$. (b) $a(t) = 2$. (c) $t = 2$.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

   (a) Use the data in the table to estimate the value of $v'(16)$.

   (b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

   (c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

   (d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

4.3 Rates of Change in Applied Contexts Other Than Motion

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS marginal cost 边际成本

Objective

Build the skills to answer exam questions on **rates of change in applied contexts other than motion**.

You must be able to:

- interpret a derivative as a rate in any context (e.g. **marginal cost 边际成本**)

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Rates in other contexts

A derivative measures the rate of change of **any** quantity: marginal cost $C'(x)$, a flow rate, a reaction rate. Interpret its sign and units just as for motion.

Example

If $C(x)$ is cost in dollars for x items, $C'(100) = 5$ means about \$5 to produce the 101st item.

2 Practice

2.1 State what marginal cost is, as a derivative. [1]

2.2 State the units of $C'(x)$ if C is in dollars and x in items. [1]

2.3 If $C'(100) = 5$, interpret this. [2]

3 Exam-style questions

3.1 Marginal cost is the derivative of [1]

- **A** revenue
- **B** the cost function
- **C** profit
- **D** price

3.2 If $A'(t) > 0$, the area A is [1]

- **A** shrinking
- **B** growing
- **C** constant
- **D** zero

3.3 $C(x)$ is cost in dollars for x items, and $C'(50) = 8$.

(a) State the units. [1]

(b) State what the sign means. [1]

(c) Interpret $C'(50) = 8$. [1]

Solutions

2.1 the derivative of the cost function, $C'(x)$.

2.2 dollars per item.

2.3 near $x = 100$ items, cost rises about \$5 per additional item.

3.1 **B**.

3.2 **B**.

3.3 (a) dollars per item. (b) cost is rising with output. (c) about \$8 to produce the 51st item.

4.4 Introduction to Related Rates

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS time 时间 • related rates 相关变化率

Objective

Build the skills to answer exam questions on **introduction to related rates**.

You must be able to:

- differentiate a relation with respect to **time 时间** to link two rates

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Related rates

When two quantities are related, their rates are related: differentiate the relation with respect to t (using the chain rule).

Example

$$A = \pi r^2 \;\Rightarrow\; \frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}.$$

2 Practice

2.1 State the first step in a related-rates problem. [1]

2.2 Differentiate $A = \pi r^2$ with respect to t. [2]

2.3 State what links the two rates. [1]

3 Exam-style questions

3.1 In related rates, you differentiate the relation with respect to [1]

- **A** x
- **B** time t
- **C** r
- **D** nothing

3.2 Differentiating $A = \pi r^2$ with respect to t gives [1]

- **A** $2\pi r$
- **B** $2\pi r \dfrac{dr}{dt}$
- **C** πr^2
- **D** $2r$

3.3 $V = s^3$ (a cube).

(a) Differentiate with respect to t. [1]

(b) State the chain-rule factor. [1]

(c) Write $\dfrac{dV}{dt}$. [1]

Solutions

2.1 write the relation connecting the quantities.

2.2 $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt}$.

2.3 the derivative of the relation with respect to t.

3.1 **B**.

3.2 **B**.

3.3 (a) $\dfrac{dV}{dt} = 3s^2\,\dfrac{ds}{dt}$. (b) $\dfrac{ds}{dt}$. (c) $\dfrac{dV}{dt} = 3s^2\,\dfrac{ds}{dt}$.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

AP CALCULUS BC • EXERCISE SHEET

4.5 Solving Related Rates Problems

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS related rates 相关变化率

Objective

Build the skills to answer exam questions on **solving related rates problems**.

You must be able to:

- carry out the full related-rates procedure and solve for the unknown rate

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Solving related rates

1. write the relation;
2. differentiate with respect to t;
3. substitute the known values and rates;
4. solve for the unknown rate.

Example

A circle's radius grows at $\frac{dr}{dt} = 2$ cm/s. At $r = 3$: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(3)(2) = 12\pi$ cm^2/s.

2 Practice

2.1 List the steps of a related-rates solution. [1]

2.2 A circle's r grows at 2 cm/s. At $r = 3$, find $\frac{dA}{dt}$ (with $A = \pi r^2$). [2]

2.3 State the units of $\dfrac{dA}{dt}$ here. [1]

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3 Exam-style questions

3.1 After differentiating, the next step is to [1]

- **A** stop
- **B** substitute known values and solve
- **C** integrate
- **D** graph

3.2 If $\dfrac{dr}{dt} = 2$ and $r = 3$, then $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$ equals [1]

- **A** 6π
- **B** 12π
- **C** 4π
- **D** 9π

3.3 $A = \pi r^2$, $\dfrac{dr}{dt} = 2$, $r = 3$.

(a) Write $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$. [1]

(b) Substitute. [1]

(c) State $\dfrac{dA}{dt}$. [1]

Solutions

2.1 write the relation; differentiate with respect to t; substitute; solve for the unknown rate.

2.2 $\dfrac{dA}{dt} = 2\pi(3)(2) = 12\pi\ \mathrm{cm}^2/\mathrm{s}$.

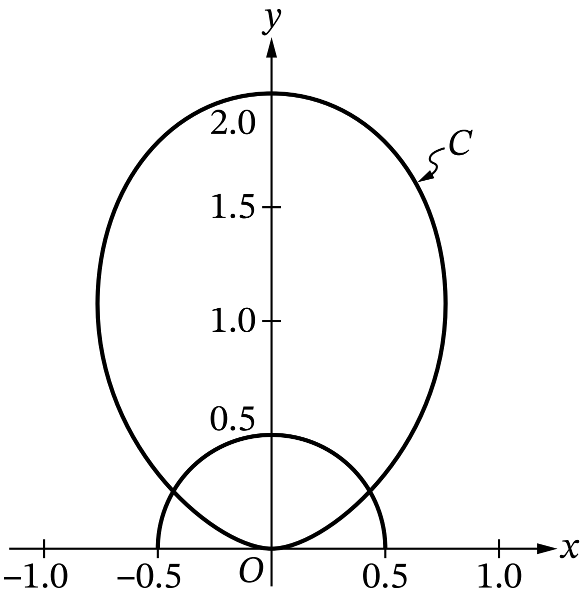
2.3 cm^2/s.

3.1 **B**.

3.2 **B**.

3.3 (a) $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt}$. (b) $2\pi(3)(2)$. (c) $12\pi\ \mathrm{cm}^2/\mathrm{s}$.

2. Curve C is defined by the polar equation $r(\theta) = 2\sin^2\theta$ for $0 \le \theta \le \pi$. Curve C and the semicircle $r = \frac{1}{2}$ for $0 \le \theta \le \pi$ are shown in the xy-plane.



(Note: Your calculator should be in radian mode.)

A. Find the rate of change of r with respect to θ at the point on curve C where $\theta = 1.3$. Show the setup for your calculations.

B. Find the area of the region that lies inside curve C but outside the graph of the polar equation $r = \frac{1}{2}$. Show the setup for your calculations.

C. It can be shown that $\frac{dx}{d\theta} = 4\sin\theta\cos^2\theta - 2\sin^3\theta$ for curve C. For $0 \le \theta \le \frac{\pi}{2}$, find the value of θ that corresponds to the point on curve C that is farthest from the y-axis. Justify your answer.

D. A particle travels along curve C so that $\frac{d\theta}{dt} = 15$ for all times t. Find the rate at which the particle's distance from the origin changes with respect to time when the particle is at the point where $\theta = 1.3$. Show the setup for your calculations.

END OF PART A

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

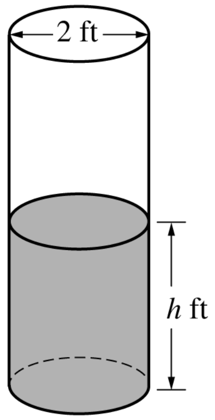
(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\, dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

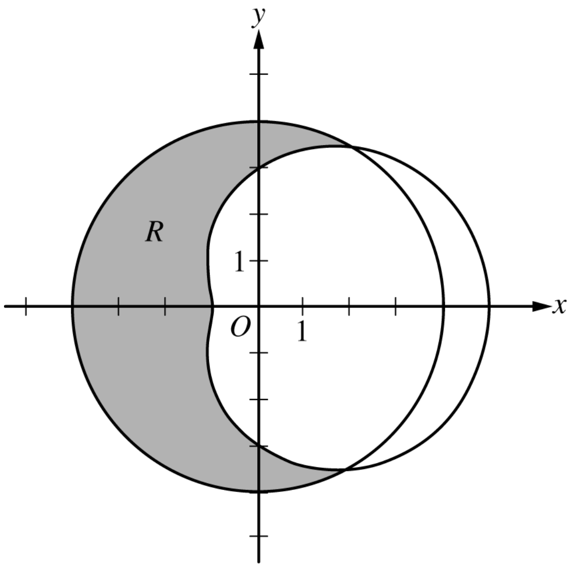
4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

   (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

   (b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

   (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

   (d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?



5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

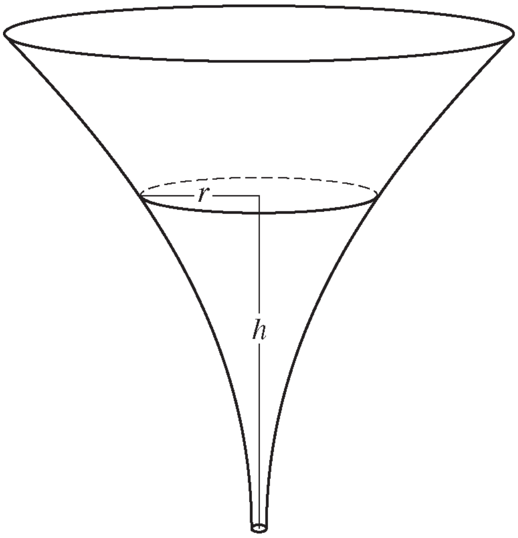
1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

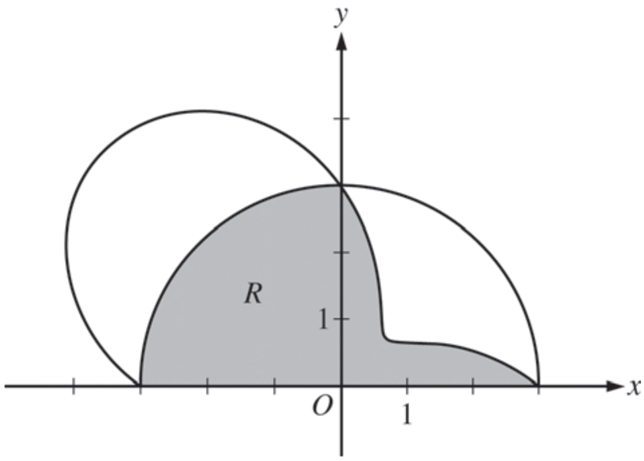


5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

   (a) Find the average value of the radius of the funnel.

   (b) Find the volume of the funnel.

   (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

   (a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

   (b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

   (c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

   (d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

END OF PART A OF SECTION II

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

AP CALCULUS BC • EXERCISE SHEET

4.6 Approximating Values Using Local Linearity and Linearization

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS linearization 线性化 • local linearity 局部线性

Objective

Build the skills to answer exam questions on **local linearity and linearization**.

You must be able to:

- write the **linearization 线性化** $L(x) = f(a) + f'(a)(x - a)$
- use it to approximate a value

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Linearization

Near $x = a$,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

— the tangent line at a.

Example

$f(x) = \sqrt{x}$ at $a = 4$: $f(4) = 2$, $f'(4) = \frac{1}{4}$, so $L(x) = 2 + \frac{1}{4}(x - 4)$ and $\sqrt{4.1} \approx 2.025$.

2 Practice

2.1 Write the linearization formula. [1]

2.2 For $f(x) = \sqrt{x}$ at $a = 4$, state $f(4)$ and $f'(4)$. [2]

2.3 State what $L(x)$ approximates. [1]

3 Exam-style questions

3.1 The linearization of f at a is [1]

- **A** $f(a)$
- **B** $f(a) + f'(a)(x - a)$
- **C** $f'(a)(x - a)$
- **D** $f(x) \cdot a$

3.2 Linearization uses the _________ line. [1]

- **A** secant
- **B** tangent
- **C** normal
- **D** horizontal

3.3 $f(x) = x^2$, $a = 3$, so $f(3) = 9$, $f'(3) = 6$.

(a) Write $L(x)$. [1]

(b) Use it for $x = 3.1$. [1]

(c) State $L(3.1)$. [1]

Solutions

2.1 $L(x) = f(a) + f'(a)(x - a)$.

2.2 $f(4) = 2$; $f'(4) = \frac{1}{4}$.

2.3 the value of $f(x)$ for x near a.

3.1 **B**.

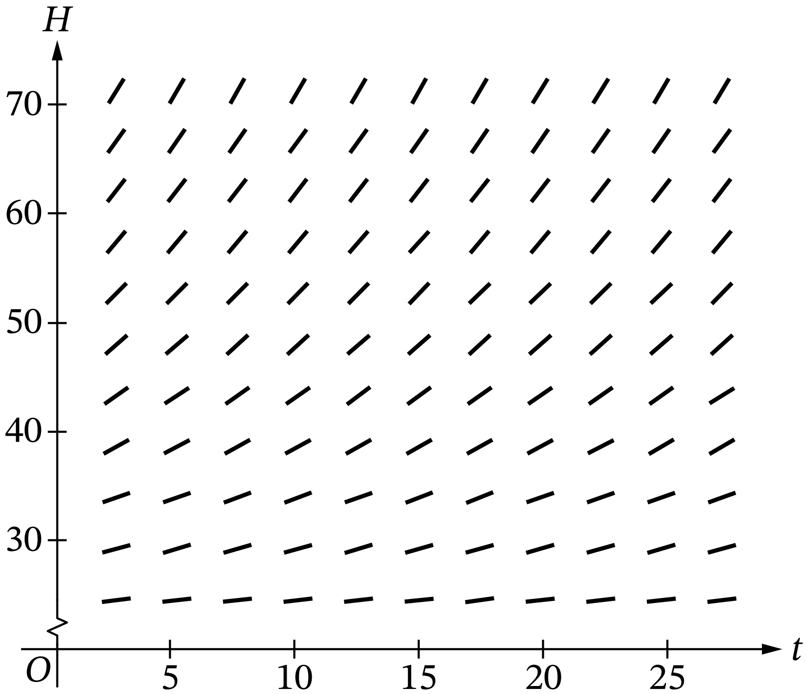
3.2 **B**.

3.3 (a) $L(x) = 9 + 6(x - 3)$. (b) $9 + 6(0.1)$. (c) 9.6.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.



Part B

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Part C

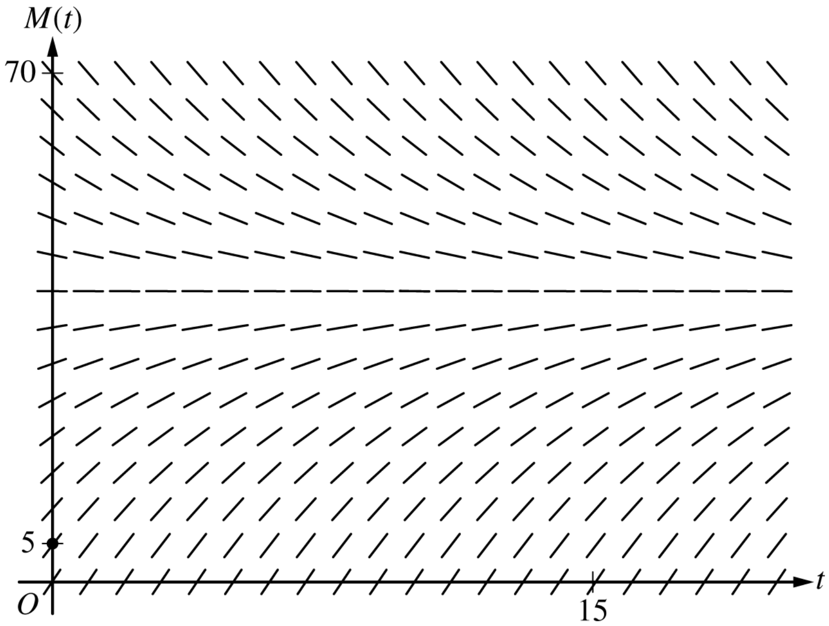
It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.



(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

   (a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

   (b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

   (c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

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5. Consider the function $f(x) = \frac{1}{x^2 - kx}$, where k is a nonzero constant. The derivative of f is given by $f'(x) = \frac{k - 2x}{\left(x^2 - kx\right)^2}$.

   (a) Let $k = 3$, so that $f(x) = \frac{1}{x^2 - 3x}$. Write an equation for the line tangent to the graph of f at the point whose x-coordinate is 4.

   (b) Let $k = 4$, so that $f(x) = \frac{1}{x^2 - 4x}$. Determine whether f has a relative minimum, a relative maximum, or neither at $x = 2$. Justify your answer.

   (c) Find the value of k for which f has a critical point at $x = -5$.

   (d) Let $k = 6$, so that $f(x) = \frac{1}{x^2 - 6x}$. Find the partial fraction decomposition for the function f. Find $\int f(x)\, dx$.

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30,$ the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t,$ where $A(t)$ is measured in pounds and t is measured in days.

   (a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30.$ Indicate units of measure.

   (b) Find the value of $A'(15).$ Using correct units, interpret the meaning of the value in the context of the problem.

   (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30.$

   (d) For $t > 30,$ $L(t),$ the linear approximation to A at $t = 30,$ is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

AP CALCULUS BC • EXERCISE SHEET

4.7 Using L'Hospital's Rule for Indeterminate Forms

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

KEY WORDS l'hospital's rule 洛必达法则 • indeterminate form 未定式

Objective

Build the skills to answer exam questions on **L'Hospital's Rule for indeterminate forms**.

You must be able to:

- recognise an **indeterminate form** 未定式 $\frac{0}{0}$ or $\frac{\infty}{\infty}$
- apply **L'Hospital's Rule** 洛必达法则

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ L'Hospital's Rule

For a limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim \frac{f}{g} = \lim \frac{f'}{g'}$$

(when the right-hand limit exists).

■ Example

$\lim_{x\to 0} \dfrac{\sin x}{x} = \lim_{x\to 0} \dfrac{\cos x}{1} = 1.$

2 Practice

2.1 State the forms L'Hospital's Rule applies to. [1]

2.2 Apply it to $\lim_{x \to 0} \dfrac{\sin x}{x}$. [2]

2.3 State what you do to the numerator and denominator. [1]

3 Exam-style questions

3.1 L'Hospital's Rule applies to [1]

- **A** any limit
- **B** $\frac{0}{0}$ or $\frac{\infty}{\infty}$
- **C** $\frac{1}{0}$
- **D** $\frac{5}{2}$

3.2 L'Hospital's Rule replaces $\dfrac{f}{g}$ with [1]

- **A** $f \cdot g$
- **B** $\dfrac{f'}{g'}$
- **C** $\dfrac{f'}{g}$
- **D** $\dfrac{f}{g'}$

3.3 $\lim_{x \to 0} \dfrac{e^x - 1}{x}$.

(a) Check the form. [1]

(b) Differentiate top and bottom. [1]

(c) Evaluate. [1]

Solutions

2.1 $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

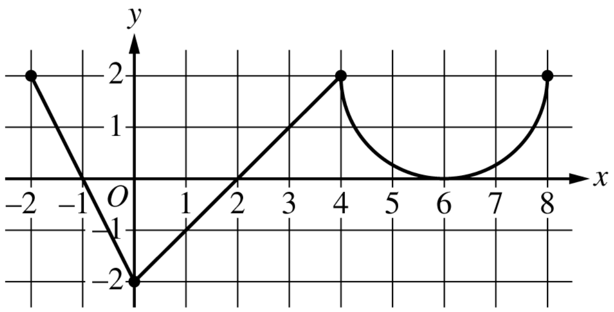
2.2 $\lim_{x\to 0} \dfrac{\cos x}{1} = 1$.

2.3 differentiate each separately, then take the limit of the new quotient.

3.1 **B**.

3.2 **B**.

3.3 (a) $\frac{0}{0}$. (b) $\dfrac{e^x}{1}$. (c) 1.



Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

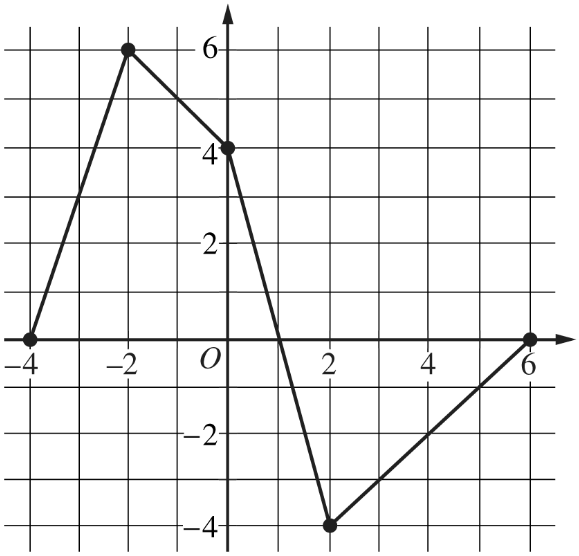
(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

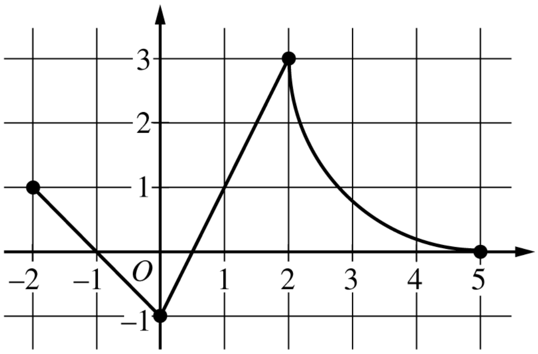
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x + 1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.