**Week 8**

# AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

## Contents 目录

# 3. Differentiation: Composite, Implicit, and Inverse Functions Starter Quiz · 课前小测

AP Calculus BC Name: ______________________ Score: ______

1. The Chain Rule gives $\dfrac{d}{dx}[f(g(x))] =$

   ☐ $f'(x) \cdot g'(x)$

   ☐ $f'(g(x)) \cdot g'(x)$

   ☐ $f'(g(x))$

   ☐ $f(g'(x))$

2. Differentiating with respect to $x$, $\dfrac{d}{dx}[y^2] =$

   ☐ $2y$

   ☐ $2y\dfrac{dy}{dx}$

   ☐ $2x$

   ☐ $2$

3. The derivative of an inverse is $\left(f^{-1}\right)'(x) =$

   ☐ $f'(x)$

   ☐ $\dfrac{1}{f'(f^{-1}(x))}$

   ☐ $\dfrac{1}{f'(x)}$

   ☐ $-f'(x)$

4. What is $\dfrac{d}{dx}[\arcsin x]$?

   ☐ $\dfrac{1}{1+x^2}$

   ☐ $\dfrac{1}{\sqrt{1-x^2}}$

   ☐ $\cos x$

   ☐ $-\dfrac{1}{\sqrt{1-x^2}}$

5. Which rule does $y = x^2 \sin x$ need at the top level?

   ☐ quotient rule

   ☐ product rule

☐ chain rule only

☐ power rule only

6. For $x^2 + y^2 = 25$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Find the slope at $(3,4)$ (a decimal).

__________________

7. If $f(2) = 5$ and $f'(2) = 4$, find $\left(f^{-1}\right)'(5)$.

__________________

8. For $f(x) = x^3 - 2x^2 + 5x$, $f'(x) = 6x - 4$. Find $f'(2)$.

__________________

9. For $y = (2x+1)^3$, find $y'$ at $x = 0$. (Chain rule: $3(2x+1)^2 \cdot 2$.)

__________________

10. The factor $\frac{dy}{dx}$ is added to every term, including pure $x$ terms like $x^2$.

☐ True ☐ False

# 3. Differentiation: Composite, Implicit, and Inverse Functions Answers · 答案

AP Calculus BC

1. $f'(g(x)) \cdot g'(x)$

   Outer derivative at the inner, times the inner derivative.

2. $2y\dfrac{dy}{dx}$

   Chain rule: $2y$ times the inner derivative $\frac{dy}{dx}$.

3. $\dfrac{1}{f'(f^{-1}(x))}$

   One over the original derivative at the matching input $f^{-1}(x)$.

4. $\dfrac{1}{\sqrt{1-x^2}}$

   arcsin has the root form $\frac{1}{\sqrt{1-x^2}}$.

5. **product rule**

   It is a product of $x^2$ and $\sin x \to$ product rule.

6. **-0.75**

   $-\frac{3}{4} = -0.75$.

7. **0.25**

   $f^{-1}(5) = 2$, so $\left(f^{-1}\right)'(5) = \frac{1}{f'(2)} = \frac{1}{4}$.

8. **8**

   $f'(2) = 6(2) - 4 = 8$.

9. **6**

   $y' = 6(2x+1)^2$; at 0: $6(1) = 6$.

10. **False**

    Only terms containing $y$ get the $\frac{dy}{dx}$ factor.

# 3.1 The Chain Rule

Name: ______ Class: ______ Date: ______ **Total: 9 marks**

**KEY WORDS** chain rule 链式法则 • composite function 复合函数

## Objective

Build the skills to answer exam questions on **the chain rule**.

**You must be able to:**

- differentiate a **composite function** **复合函数** with the **chain rule** **链式法则**

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **The chain rule**

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

— derivative of the outer (evaluated at the inner) times the derivative of the inner.

■ **Example**

$\dfrac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x = 2x\cos(x^2)$.

## 2 Practice

**2.1** State the chain rule. [1]

**2.2** Differentiate $(x^2 + 1)^3$. [2]

**2.3** State what the chain rule is for. [1]

## 3 Exam-style questions

**3.1** $\dfrac{d}{dx}[f(g(x))]$ equals [1]

- **A** $f'(x)g'(x)$
- **B** $f'(g(x)) \cdot g'(x)$
- **C** $f'(g(x))$
- **D** $g'(f(x))$

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**3.2** $\dfrac{d}{dx}[\sin(x^2)]$ equals [1]

- **A** $\cos(x^2)$
- **B** $2x\cos(x^2)$
- **C** $2x\sin(x^2)$
- **D** $\cos(2x)$

---

**3.3** $y = (3x+1)^4$.

(a) Identify the outer and inner functions. [1]

(b) Apply the chain rule. [1]

(c) State $y'$. [1]

## Solutions

**2.1** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

**2.2** $3(x^2+1)^2 \cdot 2x = 6x(x^2+1)^2$.

**2.3** differentiating a composite (a function of a function).

**3.1** **B**.

**3.2** **B**.

**3.3** (a) outer $u^4$, inner $3x+1$. (b) $4(3x+1)^3 \cdot 3$. (c) $y' = 12(3x+1)^3$.

| $t$ (seconds) | 0 | 60 | 90 | 120 | 135 | 150 |
|---|---|---|---|---|---|---|
| $f(t)$ (gallons per second) | 0 | 0.1 | 0.15 | 0.1 | 0.05 | 0 |

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function $f$, where $f(t)$ is measured in gallons per second and $t$ is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60, 90], [90, 120], and [120, 135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.

(b) Must there exist a value of $c$, for $60 < c < 120$, such that $f'(c) = 0$ ? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$.

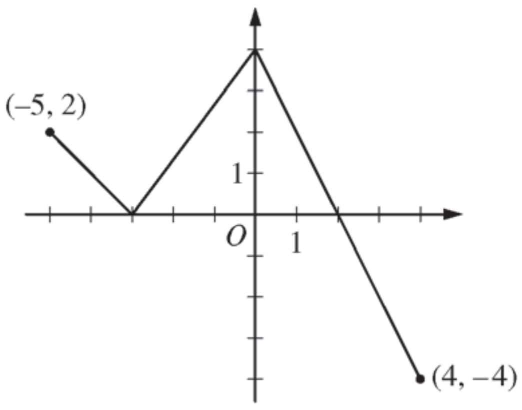
Show the setup for your calculations.

(d) Using the model $g$ defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

**CALCULUS BC**
**SECTION II, Part B**
**Time—60 minutes**
**Number of problems—4**

**No calculator is allowed for these problems.**




Graph of $f$

3. The function $f$ is defined on the closed interval $[-5, 4]$. The graph of $f$ consists of three line segments and is shown in the figure above. Let $g$ be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of $g$ both increasing and concave down? Give a reason for your answer.

   (c) The function $h$ is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function $p$ is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of $p$ at the point where $x = -1$.

# 3.2 Implicit Differentiation

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

**KEY WORDS** implicit differentiation 隐函数求导 • implicit relation 隐式关系 • chain rule 链式法则

## Objective

Build the skills to answer exam questions on **implicit differentiation**.

**You must be able to:**

- differentiate an **implicit relation 隐式关系** by attaching $\dfrac{dy}{dx}$ to every $y$
- solve for $\dfrac{dy}{dx}$

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

### Implicit differentiation

Differentiate both sides with respect to $x$, writing $y'$ each time you differentiate $y$ (the chain rule), then solve for $y'$.

### Example

$x^2 + y^2 = 25 \;\Rightarrow\; 2x + 2y\,y' = 0 \;\Rightarrow\; y' = -\dfrac{x}{y}.$

## 2 Practice

**2.1** State what to attach when differentiating $y$. [1]

**2.2** Differentiate $x^2 + y^2 = 25$ implicitly. [2]

**2.3** State the final step. [1]

## 3 Exam-style questions

**3.1** In implicit differentiation, differentiating $y^2$ gives [1]

- **A** $2y$
- **B** $2y\,y'$
- **C** $y'$
- **D** $2x$

---

**3.2** After differentiating, you [1]

- **A** stop
- **B** solve for $\dfrac{dy}{dx}$
- **C** set $y = 0$
- **D** integrate

---

**3.3** $x^2 + y^2 = 25$.

(a) Differentiate both sides. [1]

(b) Isolate the $y'$ term. [1]

(c) Solve for $y'$. [1]

## Solutions

**2.1** the factor $\dfrac{dy}{dx}$ (i.e. $y'$).

**2.2** $2x + 2y\,y' = 0$, so $y' = -\dfrac{x}{y}$.

**2.3** solve the equation for $\dfrac{dy}{dx}$.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) $2x + 2y\,y' = 0$. (b) $2y\,y' = -2x$. (c) $y' = -\dfrac{x}{y}$.

**5.** Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

**A.** Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

**B.** Write the second-degree Taylor polynomial for $f$ about $x = 1$.

**C.** The second-degree Taylor polynomial for $f$ about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all $x$ in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

**D.** Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

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# 3.3 Differentiating Inverse Functions

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

**KEY WORDS** reciprocal 倒数 • inverse function 反函数

## Objective

Build the skills to answer exam questions on **differentiating inverse functions**.

**You must be able to:**

- apply $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Derivative of an inverse**

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

The slope of the **inverse function 反函数** is the reciprocal of the slope of $f$ at the corresponding point.

■ **Example**

If $f(2) = 5$ and $f'(2) = 4$, then $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

## 2 Practice

**2.1** State the inverse-derivative formula. [1]

**2.2** If $f'(2) = 4$ and $f(2) = 5$, find $(f^{-1})'(5)$. [2]

**2.3** State the relationship between the two slopes. [1]

## 3 Exam-style questions

**3.1** $(f^{-1})'(a)$ equals [1]

- **A** $f'(a)$
- **B** $\dfrac{1}{f'(f^{-1}(a))}$
- **C** $f'(f^{-1}(a))$
- **D** $\dfrac{1}{a}$

**3.2** The slope of $f^{-1}$ is the _________ of the slope of $f$. [1]

- **A** same
- **B** reciprocal
- **C** negative
- **D** square

**3.3** $f(3) = 7$, $f'(3) = 2$.

(a) Find $f^{-1}(7)$. [1]

(b) Apply the formula. [1]

(c) Find $(f^{-1})'(7)$. [1]

## Solutions

**2.1** $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.

**2.2** $f^{-1}(5) = 2$, so $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

**2.3** they are reciprocals of each other.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) 3. (b) $\dfrac{1}{f'(3)}$. (c) $\dfrac{1}{2}$.

# 3.4 Differentiating Inverse Trigonometric Functions

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

**KEY WORDS** inverse trigonometric functions 反三角函数

## Objective

Build the skills to answer exam questions on **differentiating inverse trigonometric functions**.

**You must be able to:**

- state and use the derivatives of $\arcsin x$, $\arccos x$, and $\arctan x$

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Inverse trig derivatives**

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}.$$

## 2 Practice

**2.1** State $\dfrac{d}{dx}[\arcsin x]$. [1]

**2.2** State $\dfrac{d}{dx}[\arctan x]$. [1]

**2.3** State $\dfrac{d}{dx}[\arccos x]$. [2]

## 3 Exam-style questions

**3.1** $\dfrac{d}{dx}[\arctan x]$ equals [1]

- **A** $\dfrac{1}{\sqrt{1-x^2}}$
- **B** $\dfrac{1}{1+x^2}$
- **C** $-\dfrac{1}{1+x^2}$
- **D** $\dfrac{1}{x}$

**3.2** $\dfrac{d}{dx}[\arcsin x]$ equals [1]

- **A** $\dfrac{1}{1+x^2}$
- **B** $\dfrac{1}{\sqrt{1-x^2}}$
- **C** $-\dfrac{1}{\sqrt{1-x^2}}$
- **D** $\arccos x$

**3.3** Differentiate each.

(a) $\arcsin x$. [1]

(b) $\arctan x$. [1]

(c) $\arccos x$. [1]

## Solutions

**2.1** $\dfrac{1}{\sqrt{1-x^2}}$.

**2.2** $\dfrac{1}{1+x^2}$.

**2.3** $-\dfrac{1}{\sqrt{1-x^2}}$.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) $\dfrac{1}{\sqrt{1-x^2}}$. (b) $\dfrac{1}{1+x^2}$. (c) $-\dfrac{1}{\sqrt{1-x^2}}$.

# 3.5 Selecting Procedures for Calculating Derivatives

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

**KEY WORDS** chain rule 链式法则

## Objective

Build the skills to answer exam questions on **selecting procedures for calculating derivatives**.

**You must be able to:**

- recognise the structure of an expression and pick the right rule (power, product, quotient, chain)

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Choosing a rule**

- a power of $x$ → **power rule**;
- a product → **product rule**;
- a quotient → **quotient rule**;
- a composite (function of a function) → **chain rule**.

Combine them when an expression mixes structures.

## 2 Practice

**2.1** State which rule to use for a composite. [1]

**2.2** State which rule to use for a product. [1]

**2.3** For $y = x^2 \sin x$, name the rule. [2]

## 3 Exam-style questions

**3.1** For $y = (x^2 + 1)^5$, use the [1]

- **A** quotient rule
- **B** chain rule
- **C** product rule only
- **D** no rule

**3.2** For $y = \dfrac{x}{x+1}$, use the [1]

- **A** power rule
- **B** quotient rule
- **C** chain rule
- **D** sum rule

**3.3** Name the rule for each.

(a) $y = x^3$. [1]

(b) $y = x\,e^x$. [1]

(c) $y = \sin(x^2)$. [1]

## Solutions

**2.1** the chain rule.

**2.2** the product rule.

**2.3** the product rule.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) power rule. (b) product rule. (c) chain rule.

# 3.6 Calculating Higher-Order Derivatives

Name: ______________ Class: __________ Date: ____________ **Total: 9 marks**

**KEY WORDS** second derivative 二阶导数

## Objective

Build the skills to answer exam questions on **higher-order derivatives**.

**You must be able to:**

- find the **second derivative** 二阶导数 $f''$
- use the notation $\dfrac{d^2y}{dx^2}$

## 1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Higher-order derivatives**

The second derivative $f'' = (f')'$ is the derivative of the first derivative. Notation: $f''(x)$, $\dfrac{d^2y}{dx^2}$.

■ **Example**

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$.

## 2 Practice

**2.1** State what the second derivative is. [1]

**2.2** For $f(x) = x^3$, find $f''$. [2]

**2.3** State the notation for the second derivative. [1]

## 3 Exam-style questions

**3.1** The second derivative is the derivative of [1]

- **A** $f$
- **B** $f'$
- **C** the integral
- **D** a constant

**3.2** $\dfrac{d^2y}{dx^2}$ denotes the [1]

- **A** first derivative
- **B** second derivative
- **C** integral
- **D** slope only

**3.3** $f(x) = x^4$.

(a) Find $f'$. [1]

(b) Find $f''$. [1]

(c) State $f''(1)$. [1]

## Solutions

**2.1** the derivative of the first derivative.

**2.2** $f' = 3x^2$, so $f'' = 6x$.

**2.3** $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

**3.1** **B**.

**3.2** **B**.

**3.3** (a) $4x^3$. (b) $12x^2$. (c) 12.

**5.** Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

**A.** Find $f''(1)$, the value of $\dfrac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.

**B.** Write the second-degree Taylor polynomial for $f$ about $x = 1$.

**C.** The second-degree Taylor polynomial for $f$ about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all $x$ in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

**D.** Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

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6. The function $f$ has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of $f$ with respect to $x$. Write the fourth-degree Taylor polynomial for $f$ about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for $f$ about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\dfrac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let $g$ be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for $g$ about $x = 0$.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

**STOP**

**END OF EXAM**



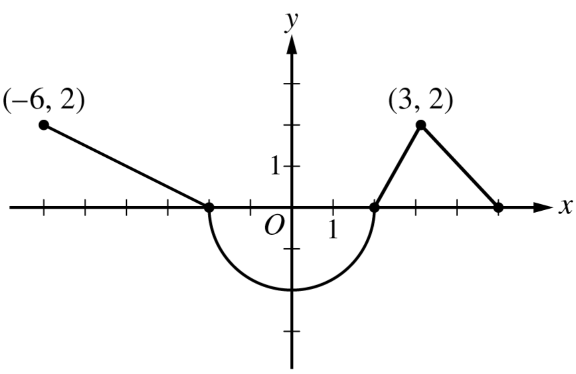
**CALCULUS BC**

**SECTION II, Part B**

**Time—1 hour**

**Number of questions—4**

**NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.**




Graph of $f'$

3. The function $f$ is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of $f'$, the derivative of $f$, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is $f$ increasing? Justify your answer.

(c) Find the absolute minimum value of $f$ on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

   (a) Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$.

   (b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of $f$ have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$ ? Justify your answer.

   (c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x + 1)^2}\right)$. Show the work that leads to your answer.

   (d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.