

Week 4

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

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3. Differentiation: Composite, Implicit, and Inverse Functions Starter Quiz · 课前小测

AP Calculus BC

Name: _____ Score: _____

- The Chain Rule gives $\frac{d}{dx}[f(g(x))]$ =
☐ $f'(x) \cdot g'(x)$
☐ $f'(g(x)) \cdot g'(x)$
☐ $f'(g(x))$
☐ $f(g'(x))$
- Differentiating with respect to x , $\frac{d}{dx}[y^2]$ =
☐ $2y$
☐ $2y \frac{dy}{dx}$
☐ $2x$
☐ 2
- The derivative of an inverse is $(f^{-1})'(x)$ =
☐ $f'(x)$
☐ $\frac{1}{f'(f^{-1}(x))}$
☐ $\frac{1}{f'(x)}$
☐ $-f'(x)$
- What is $\frac{d}{dx}[\arcsin x]$?
☐ $\frac{1}{1+x^2}$
☐ $\frac{1}{\sqrt{1-x^2}}$
☐ $\cos x$
☐ $-\frac{1}{\sqrt{1-x^2}}$
- Which rule does $y = x^2 \sin x$ need at the top level?
☐ quotient rule
☐ product rule

- ☐ chain rule only
- ☐ power rule only

6. For $x^2 + y^2 = 25$, $\frac{dy}{dx} = -\frac{x}{y}$. Find the slope at (3, 4) (a decimal).

7. If $f(2) = 5$ and $f'(2) = 4$, find $(f^{-1})'(5)$.

8. For $f(x) = x^3 - 2x^2 + 5x$, $f'(x) = 6x - 4$. Find $f'(2)$.

9. For $y = (2x + 1)^3$, find y' at $x = 0$. (Chain rule: $3(2x + 1)^2 \cdot 2$.)

10. The factor $\frac{dy}{dx}$ is added to every term, including pure x terms like x^2 .

- ☐ True ☐ False

3. Differentiation: Composite, Implicit, and Inverse Functions Answers · 答案

AP Calculus BC

1. $f'(g(x)) \cdot g'(x)$

Outer derivative at the inner, times the inner derivative.

2. $2y \frac{dy}{dx}$

Chain rule: $2y$ times the inner derivative $\frac{dy}{dx}$.

3. $\frac{1}{f'(f^{-1}(x))}$

One over the original derivative at the matching input $f^{-1}(x)$.

4. $\frac{1}{\sqrt{1-x^2}}$

arcsin has the root form $\frac{1}{\sqrt{1-x^2}}$.

5. **product rule**

It is a product of x^2 and $\sin x \rightarrow$ product rule.

6. **-0.75**

$-\frac{3}{4} = -0.75$.

7. **0.25**

$f^{-1}(5) = 2$, so $(f^{-1})'(5) = \frac{1}{f'(2)} = \frac{1}{4}$.

8. **8**

$f'(2) = 6(2) - 4 = 8$.

9. **6**

$y' = 6(2x + 1)^2$; at 0: $6(1) = 6$.

10. **False**

Only terms containing y get the $\frac{dy}{dx}$ factor.

3.1 The Chain Rule

Name: _____ Class: _____ Date: _____ Total: 9 marks

KEY WORDS chain rule 链式法则 • composite function 复合函数

Objective

Build the skills to answer exam questions on **the chain rule**.

You must be able to:

- differentiate a **composite function** 复合函数 with the **chain rule** 链式法则

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ The chain rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

— derivative of the outer (evaluated at the inner) times the derivative of the inner.

■ Example

$$\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x = 2x \cos(x^2).$$

2 Practice

2.1 State the chain rule. [1]

2.2 Differentiate $(x^2 + 1)^3$. [2]

2.3 State what the chain rule is for. [1]

3 Exam-style questions

3.1 $\frac{d}{dx}[f(g(x))]$ equals [1]

- A $f'(x)g'(x)$
- B $f'(g(x)) \cdot g'(x)$
- C $f'(g(x))$
- D $g'(f(x))$

3.2 $\frac{d}{dx}[\sin(x^2)]$ equals [1]

- A $\cos(x^2)$
- B $2x \cos(x^2)$
- C $2x \sin(x^2)$
- D $\cos(2x)$

3.3 $y = (3x + 1)^4$.

(a) Identify the outer and inner functions. [1]

(b) Apply the chain rule. [1]

(c) State y' . [1]

Solutions

2.1 $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$

2.2 $3(x^2 + 1)^2 \cdot 2x = 6x(x^2 + 1)^2.$

2.3 differentiating a composite (a function of a function).

3.1 B.

3.2 B.

3.3 (a) outer u^4 , inner $3x + 1$. (b) $4(3x + 1)^3 \cdot 3$. (c) $y' = 12(3x + 1)^3.$

Name: _____

AP Calculus BC: 2023

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f , where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t) \, dt$ in the context of the problem. Use a right

Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of

$$\int_{60}^{135} f(t) \, dt.$$

(b) Must there exist a value of c , for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right)$ for

$0 \leq t \leq 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \leq t \leq 150$.

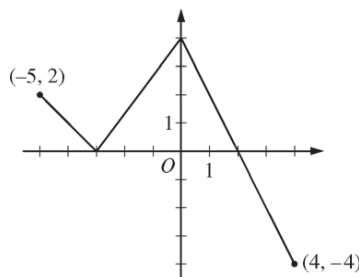
Show the setup for your calculations.

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^x f(t) dt$.
- Find $g(3)$.
 - On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.
 - The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.
 - The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

3.2 Implicit Differentiation

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS implicit differentiation 隐函数求导 • implicit relation 隐式关系
 • chain rule 链式法则

Objective

Build the skills to answer exam questions on **implicit differentiation**.

You must be able to:

- differentiate an **implicit relation** 隐式关系 by attaching $\frac{dy}{dx}$ to every y
- solve for $\frac{dy}{dx}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Implicit differentiation

Differentiate both sides with respect to x , writing y' each time you differentiate y (the chain rule), then solve for y' .

■ Example

$$x^2 + y^2 = 25 \Rightarrow 2x + 2y y' = 0 \Rightarrow y' = -\frac{x}{y}.$$

2 Practice

2.1 State what to attach when differentiating y . [1]

2.2 Differentiate $x^2 + y^2 = 25$ implicitly. [2]

2.3 State the final step. [1]

3 Exam-style questions

3.1 In implicit differentiation, differentiating y^2 gives [1]

- A $2y$
- B $2y y'$
- C y'
- D $2x$

3.2 After differentiating, you [1]

- A stop
- B solve for $\frac{dy}{dx}$
- C set $y = 0$
- D integrate

3.3 $x^2 + y^2 = 25$.

(a) Differentiate both sides. [1]

(b) Isolate the y' term. [1]

(c) Solve for y' . [1]

Solutions

2.1 the factor $\frac{dy}{dx}$ (i.e. y').

2.2 $2x + 2y y' = 0$, so $y' = -\frac{x}{y}$.

2.3 solve the equation for $\frac{dy}{dx}$.

3.1 B.

3.2 B.

3.3 (a) $2x + 2y y' = 0$. (b) $2y y' = -2x$. (c) $y' = -\frac{x}{y}$.

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.
- A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.
- B. Write the second-degree Taylor polynomial for f about $x = 1$.
- C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.
- D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.



3.3 Differentiating Inverse Functions

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS reciprocal 倒数 • inverse function 反函数

Objective

Build the skills to answer exam questions on **differentiating inverse functions**.

You must be able to:

- apply $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Derivative of an inverse

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

The slope of the **inverse function** 反函数 is the reciprocal of the slope of f at the corresponding point.

■ Example

If $f(2) = 5$ and $f'(2) = 4$, then $(f^{-1})'(5) = \frac{1}{f'(2)} = \frac{1}{4}$.

2 Practice

2.1 State the inverse-derivative formula. [1]

2.2 If $f'(2) = 4$ and $f(2) = 5$, find $(f^{-1})'(5)$. [2]

2.3 State the relationship between the two slopes. [1]

3 Exam-style questions

3.1 $(f^{-1})'(a)$ equals [1]

- **A** $f'(a)$
- **B** $\frac{1}{f'(f^{-1}(a))}$
- **C** $f'(f^{-1}(a))$
- **D** $\frac{1}{a}$

3.2 The slope of f^{-1} is the _____ of the slope of f . [1]

- **A** same
- **B** reciprocal
- **C** negative
- **D** square

3.3 $f(3) = 7$, $f'(3) = 2$.

(a) Find $f^{-1}(7)$. [1]

(b) Apply the formula. [1]

(c) Find $(f^{-1})'(7)$. [1]

Solutions

2.1 $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$

2.2 $f^{-1}(5) = 2$, so $(f^{-1})'(5) = \frac{1}{f'(2)} = \frac{1}{4}.$

2.3 they are reciprocals of each other.

3.1 **B.**

3.2 **B.**

3.3 (a) 3. (b) $\frac{1}{f'(3)}$. (c) $\frac{1}{2}.$

3.4 Differentiating Inverse Trigonometric Functions

Name: _____ Class: _____ Date: _____ Total: 9 marks

KEY WORDS inverse trigonometric functions 反三角函数

Objective

Build the skills to answer exam questions on **differentiating inverse trigonometric functions**.

You must be able to:

- state and use the derivatives of $\arcsin x$, $\arccos x$, and $\arctan x$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Inverse trig derivatives

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}.$$

2 Practice

2.1 State $\frac{d}{dx}[\arcsin x]$. [1]

2.2 State $\frac{d}{dx}[\arctan x]$. [1]

2.3 State $\frac{d}{dx}[\arccos x]$. [2]

3 Exam-style questions

3.1 $\frac{d}{dx}[\arctan x]$ equals [1]

- A $\frac{1}{\sqrt{1-x^2}}$
- B $\frac{1}{1+x^2}$
- C $-\frac{1}{1+x^2}$
- D $\frac{1}{x}$

3.2 $\frac{d}{dx}[\arcsin x]$ equals [1]

- A $\frac{1}{1+x^2}$
- B $\frac{1}{\sqrt{1-x^2}}$
- C $-\frac{1}{\sqrt{1-x^2}}$
- D $\arccos x$

3.3 Differentiate each.

(a) $\arcsin x$. [1]

(b) $\arctan x$. [1]

(c) $\arccos x$. [1]

Solutions

2.1 $\frac{1}{\sqrt{1-x^2}}.$

2.2 $\frac{1}{1+x^2}.$

2.3 $-\frac{1}{\sqrt{1-x^2}}.$

3.1 B.

3.2 B.

3.3 (a) $\frac{1}{\sqrt{1-x^2}}.$ (b) $\frac{1}{1+x^2}.$ (c) $-\frac{1}{\sqrt{1-x^2}}.$

3.5 Selecting Procedures for Calculating Derivatives

Name: _____ Class: _____ Date: _____ Total: 9 marks

KEY WORDS chain rule 链式法则

Objective

Build the skills to answer exam questions on **selecting procedures for calculating derivatives**.

You must be able to:

- recognise the structure of an expression and pick the right rule (power, product, quotient, chain)

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Choosing a rule

- a power of $x \rightarrow$ **power rule**;
- a product $\rightarrow$ **product rule**;
- a quotient $\rightarrow$ **quotient rule**;
- a composite (function of a function) $\rightarrow$ **chain rule**.

Combine them when an expression mixes structures.

2 Practice

2.1 State which rule to use for a composite. [1]

2.2 State which rule to use for a product. [1]

2.3 For $y = x^2 \sin x$, name the rule. [2]

3 Exam-style questions

3.1 For $y = (x^2 + 1)^5$, use the [1]

- A quotient rule
- B chain rule
- C product rule only
- D no rule

3.2 For $y = \frac{x}{x+1}$, use the [1]

- A power rule
- B quotient rule
- C chain rule
- D sum rule

3.3 Name the rule for each.

(a) $y = x^3$. [1]

(b) $y = x e^x$. [1]

(c) $y = \sin(x^2)$. [1]

Solutions

2.1 the chain rule.

2.2 the product rule.

2.3 the product rule.

3.1 B.

3.2 B.

3.3 (a) power rule. (b) product rule. (c) chain rule.

3.6 Calculating Higher-Order Derivatives

Name: _____ Class: _____ Date: _____

Total: 9 marks

KEY WORDS second derivative 二阶导数

Objective

Build the skills to answer exam questions on **higher-order derivatives**.

You must be able to:

- find the **second derivative** 二阶导数 f''
- use the notation $\frac{d^2y}{dx^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Higher-order derivatives

The second derivative $f'' = (f')'$ is the derivative of the first derivative. Notation:

$$f''(x), \frac{d^2y}{dx^2}.$$

■ Example

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x.$$

2 Practice

2.1 State what the second derivative is. [1]

2.2 For $f(x) = x^3$, find f'' . [2]

2.3 State the notation for the second derivative. [1]

3 Exam-style questions

3.1 The second derivative is the derivative of [1]

- **A** f
 - **B** f'
 - **C** the integral
 - **D** a constant
-

3.2 $\frac{d^2y}{dx^2}$ denotes the [1]

- **A** first derivative
 - **B** second derivative
 - **C** integral
 - **D** slope only
-

3.3 $f(x) = x^4$.

(a) Find f' . [1]

(b) Find f'' . [1]

(c) State $f''(1)$. [1]

Solutions

2.1 the derivative of the first derivative.

2.2 $f' = 3x^2$, so $f'' = 6x$.

2.3 $f''(x)$ or $\frac{d^2y}{dx^2}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $4x^3$. (b) $12x^2$. (c) 12.

5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

- A. Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.
- B. Write the second-degree Taylor polynomial for f about $x = 1$.
- C. The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \leq 60$ for all x in the interval $1 \leq x \leq 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.
- D. Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

AP Calculus BC: 2023

6. The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$,

$$f''(x) = -f(x^2), \text{ and } f'''(x) = -2x \cdot f'(x^2).$$

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x . Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that

$$|f^{(5)}(x)| \leq 15 \text{ for } 0 \leq x \leq 0.5, \text{ use the Lagrange error bound to show that this approximation is within } \frac{1}{10^5} \text{ of the exact value of } f(0.1).$$

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.



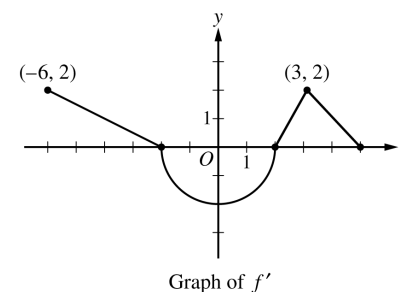
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END OF EXAM

AP Calculus BC: 2017

CALCULUS BC
SECTION II, Part B
Time—1 hour
Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f' , the derivative of f , consists of a semicircle and three line segments, as shown in the figure above.

- (a) Find the values of $f(-6)$ and $f(5)$.
- (b) On what intervals is f increasing? Justify your answer.
- (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.
- (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.



4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y .

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find

$\lim_{x \rightarrow -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.
