Week 2

AP Calculus BC

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

2. Differentiation: Definition and Fundamental Properties Starter Quiz · 课前小测

AP Calculus BC   Name: ______________   Score: ______

1. Find the average rate of change of $f(x) = x^2$ on $[1, 3]$.

   ____

2. Which expression is the limit definition of $f'(x)$?

   - ☐ $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
   - ☐ $\dfrac{f(x+h)-f(x)}{h}$ with $h = 0$
   - ☐ $\lim_{h\to 0} \big(f(x+h)-f(x)\big)$
   - ☐ $f(x+h)-f(x)$

3. A table gives $f(1.9) = 4.2$ and $f(2.1) = 5.8$. Use the symmetric difference quotient to estimate $f'(2)$.

   ____

4. Which implication is **true**?

   - ☐ continuous at $c \Rightarrow$ differentiable at c
   - ☐ differentiable at $c \Rightarrow$ continuous at c
   - ☐ discontinuous at $c \Rightarrow$ differentiable at c
   - ☐ neither implies anything

5. What is $\dfrac{d}{dx}[x^5]$?

   - ☐ $5x^4$
   - ☐ x^4
   - ☐ $5x^6$
   - ☐ $4x^5$

6. What is $\dfrac{d}{dx}[12]$?

   ____

7. What is $\dfrac{d}{dx}[\sin x]$?

   - ☐ $\cos x$
   - ☐ $-\cos x$

   - ☐ $-\sin x$
   - ☐ $\sin x$

8. The derivative of $f \cdot g$ equals $f' \cdot g'$.

   ☐ True   ☐ False

9. The Quotient Rule numerator for $\dfrac{f}{g}$ is...

   - ☐ $f'g'$
   - ☐ $f'g - fg'$
   - ☐ $fg' - f'g$
   - ☐ $f'g + fg'$

10. You can compute $f'(x)$ by setting $h = 0$ directly in $\dfrac{f(x+h)-f(x)}{h}$.

    ☐ True   ☐ False

AP Calculus BC

1. **4**
   $\frac{9-1}{3-1} = \frac{8}{2} = 4.$
2. $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
   The derivative is the limit of the difference quotient as $h \to 0$.
3. **8**
   $\frac{5.8-4.2}{2(0.1)} = \frac{1.6}{0.2} = 8.$
4. **differentiable at $c \Rightarrow$ continuous at c**
   Differentiable forces continuous; the reverse is false.
5. $5x^4$
   Bring down 5, reduce exponent: $5x^{5-1} = 5x^4$.
6. **0**
   The derivative of any constant is 0.
7. $\cos x$
   $\frac{d}{dx}[\sin x] = \cos x$.
8. **False**
   It is $f'g + fg'$, not $f'g'$.
9. $f'g - fg'$
   Numerator is $f'g - fg'$, over g^2.
10. **False**
    That gives $\frac{0}{0}$; you must take the limit after simplifying.

AP CALCULUS BC • EXERCISE SHEET

2.1 Average and Instantaneous Rates of Change at a Point

Name: ________ Class: ________ Date: ________ **Total: 9 marks**

KEY WORDS instantaneous 瞬时 • difference quotient 差商 • derivative 导数 • slope 斜率

Objective

Build the skills to answer exam questions on **average and instantaneous rates of change at a point.**

You must be able to:

- form the **difference quotient** 差商 for the average rate of change
- describe the instantaneous rate as its limit

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Average and instantaneous rates

Average rate over $[a, a+h]$ is the **secant slope** $\dfrac{f(a+h)-f(a)}{h}$.

The instantaneous rate at a is the **tangent slope**, obtained as $h \to 0$.

Example

$f(x) = x^2$ at $a = 3$: $\dfrac{(3+h)^2 - 9}{h} = 6 + h \to 6$ as $h \to 0$.

2 Practice

2.1 State the difference quotient for the average rate over $[a, a+h]$. [1]

2.2 State what happens to it as $h \to 0$. [1]

2.3 For $f(x) = x^2$, compute $\dfrac{f(2+h) - f(2)}{h}$ and simplify. [2]

3 Exam-style questions

3.1 The average rate of change over $[a, a+h]$ is [1]

- **A** $f(a)h$
- **B** $\dfrac{f(a+h) - f(a)}{h}$
- **C** $f'(a)$
- **D** h

3.2 The instantaneous rate is the average rate as [1]

- **A** $h \to \infty$
- **B** $h \to 0$
- **C** $h = 1$
- **D** $a \to 0$

3.3 $f(x) = x^2$ at $a = 3$.

(a) Write $f(3+h)$. [1]

(b) Form the difference quotient. [1]

(c) State its limit as $h \to 0$. [1]

Solutions

2.1 $\dfrac{f(a+h) - f(a)}{h}$.

2.2 it approaches the instantaneous rate of change (the derivative).

2.3 $\dfrac{(2+h)^2 - 4}{h} = \dfrac{4h + h^2}{h} = 4 + h$.

3.1 **B**.

3.2 **B**.

3.3 (a) $(3+h)^2$. (b) $\dfrac{(3+h)^2 - 9}{h} = 6 + h$. (c) 6.

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

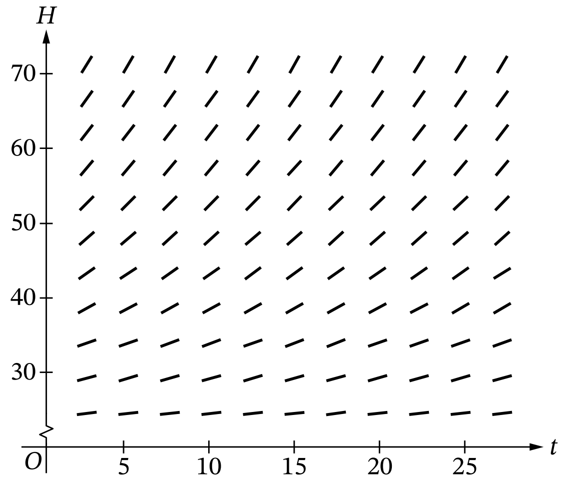
Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

3. A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

Part A

Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.



Part B

Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Part C

It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Part D

Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

1. An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

A. Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

B. Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

C. Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

D. At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\,dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

AP CALCULUS BC • EXERCISE SHEET

2.2 Defining the Derivative and Reading Its Notation

Name: ________ Class: ________ Date: ________ **Total: 9 marks**

KEY WORDS derivative 导数 • notations 记号 • slope 斜率 • tangent line 切线 • instantaneous 瞬时

Objective

Build the skills to answer exam questions on **defining the derivative and using derivative notation**.

You must be able to:

- write the limit definition of the **derivative** 导数
- use the notations $f'(x)$ and $\dfrac{dy}{dx}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **The derivative**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is the instantaneous rate of change — the slope of the **tangent line** 切线. Notations: $f'(x)$, $\dfrac{dy}{dx}$, $\dfrac{d}{dx}[f(x)]$.

2 Practice

2.1 Write the limit definition of the derivative. [1]

2.2 State two notations for the derivative. [1]

2.3 State what $f'(x)$ measures geometrically. [2]

3 Exam-style questions

3.1 The derivative $f'(x)$ is defined as [1]

- **A** $f(x)/x$
- **B** $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$
- **C** $f(x) \cdot x$
- **D** $\dfrac{f(b) - f(a)}{b - a}$ only

3.2 $\dfrac{dy}{dx}$ is another notation for the [1]

- **A** integral
- **B** derivative
- **C** limit only
- **D** area

3.3 Answer each.

(a) $f'(a)$ is the slope of the ________ line. [1]

(b) Name the quantity $f'(x)$. [1]

(c) State what $\dfrac{dy}{dx}$ reads as. [1]

Solutions

2.1 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

2.2 any two of $f'(x)$, $\dfrac{dy}{dx}$, $\dfrac{d}{dx}[f(x)]$.

2.3 the slope of the tangent line to the graph of f at x.

3.1 **B**.

3.2 **B**.

3.3 (a) tangent. (b) the derivative (instantaneous rate of change). (c) the rate of change of y with respect to x.

AP CALCULUS BC • EXERCISE SHEET

2.3 Estimating a Derivative at a Point

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS table 表格 • difference quotient 差商 • derivative 导数 • slope 斜率

Objective

Build the skills to answer exam questions on **estimating derivatives at a point**.

You must be able to:

- estimate $f'(a)$ from a **table 表格** of values using a small-h difference quotient

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Estimating a derivative**

Estimate $f'(a)$ with a small-h difference quotient from table values:

$$f'(a) \approx \frac{f(a+h)-f(a)}{h} \quad \text{or} \quad \frac{f(a+h)-f(a-h)}{2h} \text{ (symmetric).}$$

■ **Example**

$f(2)=5$, $f(2.1)=5.4$: $f'(2) \approx \dfrac{5.4-5}{0.1} = 4$.

2 Practice

2.1 State a formula to estimate $f'(a)$ from a table. [1]

2.2 If $f(2)=5$ and $f(2.1)=5.4$, estimate $f'(2)$. [2]

2.3 State why a smaller h generally gives a better estimate. [1]

3 Exam-style questions

3.1 A derivative at a point can be estimated from a table using a [1]

- **A** single value
- **B** difference quotient with small h
- **C** maximum
- **D** mean

3.2 The symmetric difference quotient is [1]

- **A** $\dfrac{f(a+h)-f(a)}{h}$
- **B** $\dfrac{f(a+h)-f(a-h)}{2h}$
- **C** $\dfrac{f(a)}{h}$
- **D** $f(a+h)\cdot f(a-h)$

3.3 $f(3)=10,\ f(3.2)=10.6.$

(a) Find Δf. [1]

(b) Find Δx. [1]

(c) Estimate $f'(3)$. [1]

Solutions

2.1 $f'(a) \approx \dfrac{f(a+h)-f(a)}{h}$ for small h.

2.2 $\dfrac{5.4-5}{2.1-2} = \dfrac{0.4}{0.1} = 4.$

2.3 a smaller interval is closer to the tangent slope (the limit).

3.1 **B**.

3.2 **B**.

3.3 (a) 0.6. (b) 0.2. (c) $\dfrac{0.6}{0.2} = 3.$

1. Male birds of a certain species arrive at a nesting area over a thirty-day period. The rate at which the male birds arrive at the nesting area at time t days is modeled by a differentiable function M, where $M(t)$ is measured in number of birds per day. Selected values of $M(t)$ are shown in the table.

t (days)	0	5	10	15	20	25	30
$M(t)$ (birds per day)	2	7	16	6	5	2	0

(Note: Your calculator should be in radian mode.)

Part A

Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$. Show the work that leads to your answer, and indicate units of measure.

Part B

i. Use a midpoint Riemann sum with the three subintervals $[0,10]$, $[10,20]$, and $[20,30]$ to approximate $\int_0^{30} M(t)\,dt$. Show the work that leads to your answer.

ii. Interpret the meaning of $\int_0^{30} M(t)\,dt$ in the context of the problem.

Part C

The rate at which female birds of the same species arrive at the same nesting area, in birds per day, is modeled by the function F defined as follows.

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t < 15 \\ 18 + 16\sin\left(\frac{\pi}{20}(t+15)\right) & \text{for } 15 \le t \le 45 \end{cases}$$

How many female birds of this species arrive at the nesting area from $t = 15$ to $t = 45$? Show the setup for your calculations, and round your answer to the nearest integer.

Part D

On the interval $15 < t < 30$, the difference in the rates at which male and female birds of this species arrive at the nesting area can be modeled by the differentiable function $D(t) = M(t) - F(t)$, where F is the function defined in part C. Is there a time t in the interval $15 < t < 20$ when $D(t) = 0$? Justify your answer.

3. A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \le t \le 7$. Show the work that leads to your answer and include units of measure.

   (b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

   (c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

   (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

4. An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

   (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

   (b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

   (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

   (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (hours)	0	1	3	6	8
$R(t)$ (liters / hour)	1340	1190	950	740	700

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where R is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time t when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	–220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

2.4 Differentiability and Continuity: When a Derivative Exists

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS differentiable 可导 • continuous 连续 • continuity 连续性 • differentiability 可导性 • corner 尖点

Objective

Build the skills to answer exam questions on **differentiability and continuity**.

You must be able to:

- use that **differentiability 可导性** implies **continuity 连续性** (but not the reverse)
- identify where a continuous function fails to be differentiable

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ Differentiability and continuity

If f is differentiable at a, then f is continuous at a. The converse **fails**: f can be continuous yet not differentiable — at a **corner**, cusp, or vertical tangent.

■ Example

$f(x) = |x|$ is continuous at 0 but has a corner there, so it is not differentiable at 0.

2 Practice

2.1 State the implication between differentiability and continuity. [1]

2.2 State whether continuity implies differentiability. [1]

2.3 Give an example of a continuous but non-differentiable point. [2]

3 Exam-style questions

3.1 If f is differentiable at a, then f is [1]

- **A** discontinuous
- **B** continuous at a
- **C** linear
- **D** constant

3.2 $f(x) = |x|$ at $x = 0$ is [1]

- **A** differentiable
- **B** continuous but not differentiable
- **C** discontinuous
- **D** undefined

3.3 Continuous functions can fail to be differentiable at certain features.

(a) Name the feature at a sharp point. [1]

(b) Name the feature with an infinite slope. [1]

(c) State whether $|x|$ is differentiable at 0. [1]

Solutions

2.1 differentiable at $a \Rightarrow$ continuous at a.

2.2 no (continuity does not imply differentiability).

2.3 $f(x) = |x|$ at $x = 0$ (a corner).

3.1 **B**.

3.2 **B**.

3.3 (a) a corner. (b) a vertical tangent. (c) no.

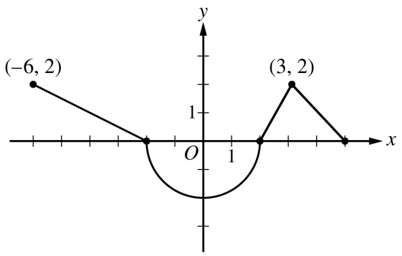
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is f increasing? Justify your answer.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

AP CALCULUS BC • EXERCISE SHEET

2.5 The Power Rule

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS power rule 幂法则

Objective

Build the skills to answer exam questions on **applying the power rule**.

You must be able to:

- differentiate a power of x with the **power rule** 幂法则 $\frac{d}{dx}[x^n] = nx^{n-1}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

The power rule

$$\frac{d}{dx}[x^n] = n\,x^{n-1}.$$

Special cases: $\frac{d}{dx}[x] = 1$ and $\frac{d}{dx}[c] = 0$ for a constant c.

Example

$\frac{d}{dx}[x^3] = 3x^2$.

2 Practice

2.1 State the power rule. [1]

2.2 Differentiate x^5. [1]

2.3 Differentiate x^2 and x. [2]

3 Exam-style questions

3.1 $\frac{d}{dx}[x^n]$ equals [1]

- **A** x^{n-1}
- **B** $n\,x^{n-1}$
- **C** $n\,x^n$
- **D** $\frac{x^{n+1}}{n+1}$

3.2 $\frac{d}{dx}[x^4]$ equals [1]

- **A** $4x^3$
- **B** x^3
- **C** $4x^5$
- **D** $3x^4$

3.3 Differentiate each.

(a) x^6. [1]

(b) x. [1]

(c) 7 (a constant). [1]

Solutions

2.1 $\frac{d}{dx}[x^n] = n\,x^{n-1}$.

2.2 $5x^4$.

2.3 $\frac{d}{dx}[x^2] = 2x$; $\frac{d}{dx}[x] = 1$.

3.1 **B**.

3.2 **A**.

3.3 (a) $6x^5$. (b) 1. (c) 0.

2.6 Constant, Sum, Difference, and Constant Multiple Rules

Name: ________ Class: ________ Date: ________ **Total: 9 marks**

KEY WORDS constant multiple 常数倍 • constant multiple rule 常数倍法则 • sum/difference rule 和差法则 • derivative 导数

Objective

Build the skills to answer exam questions on **the constant, sum, difference, and constant multiple rules**.

You must be able to:

- apply the **constant multiple rule** 常数倍法则 and **sum/difference rule** 和差法则

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **Basic derivative rules**

- **constant multiple**: $\frac{d}{dx}[c\,f] = c\,f'$;
- **sum / difference**: $\frac{d}{dx}[f \pm g] = f' \pm g'$;
- **constant**: $\frac{d}{dx}[c] = 0$.

■ **Example**

$\frac{d}{dx}[4x^3 - 2x] = 12x^2 - 2$.

2 Practice

2.1 State the sum rule. [1]

2.2 Differentiate $3x^2$. [1]

2.3 Differentiate $x^3 + x$. [2]

3 Exam-style questions

3.1 $\frac{d}{dx}[5x^2]$ equals [1]

- **A** $5x$
- **B** $10x$
- **C** $5x^2$
- **D** $10x^2$

3.2 $\frac{d}{dx}[f + g]$ equals [1]

- **A** $f' \cdot g'$
- **B** $f' + g'$
- **C** $f' - g'$
- **D** fg

3.3 $y = 4x^3 - 2x$.

(a) Differentiate $4x^3$. [1]

(b) Differentiate $-2x$. [1]

(c) State y'. [1]

Solutions

2.1 $\frac{d}{dx}[f + g] = f' + g'$.

2.2 $6x$.

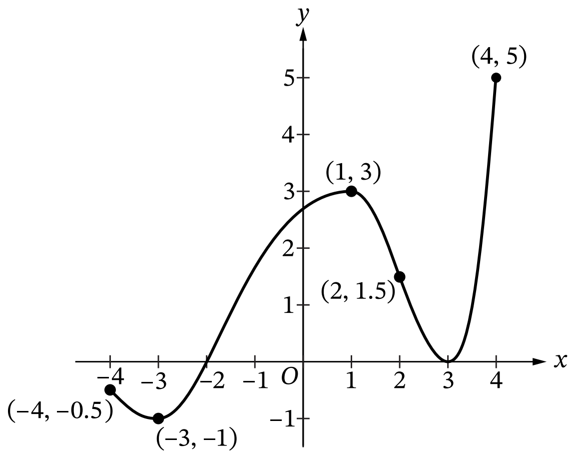
2.3 $3x^2 + 1$.

3.1 **B**.

3.2 **B**.

3.3 (a) $12x^2$. (b) -2. (c) $y' = 12x^2 - 2$.

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.



Graph of f'

Part A

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Part C

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

Part D

For $-4 \le x \le 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

AP CALCULUS BC • EXERCISE SHEET

2.7 Derivatives of cos x, sin x, e^x, and ln x

Name: ________ Class: ________ Date: ________ **Total: 9 marks**

KEY WORDS derivatives 导数

Objective

Build the skills to answer exam questions on **the derivatives of** $\cos x$**,** $\sin x$**,** e^x**, and** $\ln x$.

You must be able to:

- state and use these four standard **derivatives 导数**

1 Worked examples

Study these first. Each one shows the method for a question type used later.

Key derivatives

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x,$$

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[\ln x] = \frac{1}{x}.$$

2 Practice

2.1 State $\dfrac{d}{dx}[\sin x]$. [1]

2.2 State $\dfrac{d}{dx}[e^x]$. [1]

2.3 State $\dfrac{d}{dx}[\cos x]$ and $\dfrac{d}{dx}[\ln x]$. [2]

3 Exam-style questions

3.1 $\dfrac{d}{dx}[\cos x]$ equals [1]

- **A** $\sin x$
- **B** $-\sin x$
- **C** $\cos x$
- **D** $-\cos x$

3.2 $\dfrac{d}{dx}[\ln x]$ equals [1]

- **A** x
- **B** $\dfrac{1}{x}$
- **C** $\ln x$
- **D** e^x

3.3 Differentiate each.

(a) $\sin x$. [1]

(b) e^x. [1]

(c) $\cos x$. [1]

Solutions

2.1 $\cos x$.

2.2 e^x.

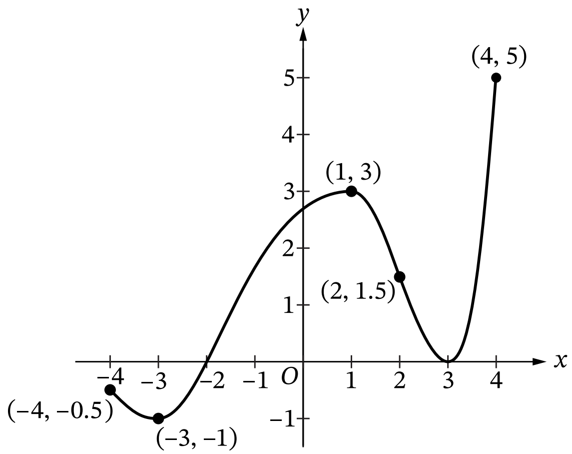
2.3 $\dfrac{d}{dx}[\cos x] = -\sin x$; $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$.

3.1 **B**.

3.2 **B**.

3.3 (a) $\cos x$. (b) e^x. (c) $-\sin x$.

4. Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.



Graph of f'

Part A

For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.

Part B

Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Part C

For $-4 \le x \le 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

Part D

For $-4 \le x \le 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

2.8 The Product Rule

Name: ____________ Class: ________ Date: __________ **Total: 9 marks**

KEY WORDS product rule 乘积法则 • power rule 幂法则 • quotient rule 商法则

Objective

Build the skills to answer exam questions on **the product rule**.

You must be able to:

- differentiate a product with the **product rule** 乘积法则 $(fg)' = f'g + fg'$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

The product rule

$$(fg)' = f'g + fg'.$$

Example

$$\frac{d}{dx}[x \sin x] = (1)\sin x + x\cos x = \sin x + x\cos x.$$

2 Practice

2.1 State the product rule. [1]

2.2 Differentiate $x\,e^{x}$. [2]

2.3 State when you need the product rule. [1]

3 Exam-style questions

3.1 $(fg)'$ equals [1]

- **A** $f'g'$
- **B** $f'g + fg'$
- **C** $f'g - fg'$
- **D** f'/g'

3.2 $\frac{d}{dx}[x^2 \sin x]$ uses the [1]

- **A** power rule only
- **B** product rule
- **C** quotient rule
- **D** chain rule only

3.3 $y = x \cos x$.

(a) Identify f and g. [1]

(b) Apply the product rule. [1]

(c) State y'. [1]

Solutions

2.1 $(fg)' = f'g + fg'$.

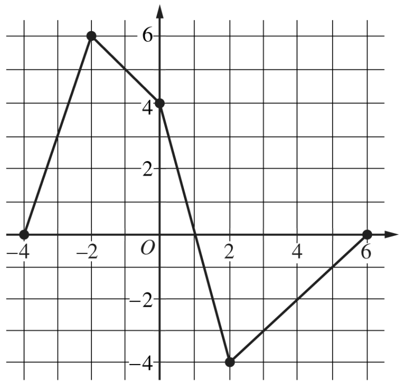
2.2 $(1)e^x + x\,e^x = e^x(1 + x)$.

2.3 when differentiating a product of two functions of x.

3.1 **B**.

3.2 **B**.

3.3 (a) $f = x$, $g = \cos x$. (b) $(1)\cos x + x(-\sin x)$. (c) $y' = \cos x - x \sin x$.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

AP CALCULUS BC • EXERCISE SHEET

2.9 The Quotient Rule

Name: ______ Class: ______ Date: ______ **Total: 9 marks**

KEY WORDS quotient rule 商法则

Objective

Build the skills to answer exam questions on **the quotient rule**.

You must be able to:

- differentiate a quotient with the **quotient rule** 商法则 $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **The quotient rule**

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

■ **Example**

$$\frac{d}{dx}\left[\frac{x}{x+1}\right] = \frac{(1)(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

2 Practice

2.1 State the quotient rule. [1]

2.2 Differentiate $\dfrac{1}{x}$ using the quotient rule. [2]

2.3 State the denominator of the quotient-rule result. [1]

3 Exam-style questions

3.1 $\left(\frac{f}{g}\right)'$ equals [1]

- **A** $\dfrac{f'g - fg'}{g^2}$
- **B** $\dfrac{f'g + fg'}{g^2}$
- **C** $\dfrac{f'}{g'}$
- **D** $\dfrac{fg' - f'g}{g^2}$

3.2 In the quotient rule, the denominator is [1]

- **A** g
- **B** g^2
- **C** f^2
- **D** fg

3.3 $y = \dfrac{x}{x+2}$.

(a) Identify f and g. [1]

(b) Apply the rule. [1]

(c) Simplify. [1]

Solutions

2.1 $\left(\frac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$.

2.2 with $f = 1$, $g = x$: $\dfrac{(0)(x) - (1)(1)}{x^2} = -\dfrac{1}{x^2}$.

2.3 g^2 (the square of the denominator).

3.1 **A**.

3.2 **B**.

3.3 (a) $f = x$, $g = x + 2$. (b) $\dfrac{(1)(x+2) - x(1)}{(x+2)^2}$. (c) $\dfrac{2}{(x+2)^2}$.

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\, dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\, dx$ or show that it diverges.

5. Let f be the function defined by $f(x) = \dfrac{[illegible]}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

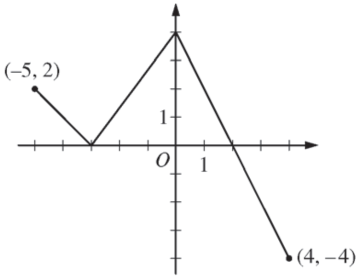
(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\, dx$ or show that the integral diverges.

(d) Determine whether the series $\sum_{n=5}^{\infty} \dfrac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

AP CALCULUS BC • EXERCISE SHEET

2.10 Derivatives of Tangent, Cotangent, Secant, and Cosecant

Name: ____________ Class: __________ Date: __________ **Total: 9 marks**

KEY WORDS trigonometric functions 三角函数 • derivative 导数

Objective

Build the skills to answer exam questions on **the derivatives of tangent, cotangent, secant, and cosecant**.

You must be able to:

- state and use the derivatives of the four remaining **trigonometric functions** 三角函数

1 Worked examples

Study these first. Each one shows the method for a question type used later.

■ **More trig derivatives**

$$\frac{d}{dx}[\tan x] = \sec^2 x, \quad \frac{d}{dx}[\cot x] = -\csc^2 x,$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x, \quad \frac{d}{dx}[\csc x] = -\csc x \cot x.$$

2 Practice

2.1 State $\dfrac{d}{dx}[\tan x]$. [1]

2.2 State $\dfrac{d}{dx}[\sec x]$. [1]

2.3 State $\dfrac{d}{dx}[\cot x]$ and $\dfrac{d}{dx}[\csc x]$. [2]

3 Exam-style questions

3.1 $\dfrac{d}{dx}[\tan x]$ equals [1]

- **A** $\sec^2 x$
- **B** $-\csc^2 x$
- **C** $\sec x \tan x$
- **D** $-\sin x$

3.2 $\dfrac{d}{dx}[\sec x]$ equals [1]

- **A** $\sec^2 x$
- **B** $\sec x \tan x$
- **C** $-\csc x \cot x$
- **D** $\tan x$

3.3 Differentiate each.

(a) $\tan x$. [1]

(b) $\cot x$. [1]

(c) $\csc x$. [1]

Solutions

2.1 $\sec^2 x$.

2.2 $\sec x \tan x$.

2.3 $\dfrac{d}{dx}[\cot x] = -\csc^2 x$; $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$.

3.1 **A**.

3.2 **B**.

3.3 (a) $\sec^2 x$. (b) $-\csc^2 x$. (c) $-\csc x \cot x$.