Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = (3 - x)y^2$ with initial condition $f(1) = -1$.

	Model Solution	Scoring	
A	Find $f''(1)$, the value of $\frac{d^2y}{dx^2}$ at the point $(1, -1)$. Show the work that leads to your answer.		
	$\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y\frac{dy}{dx}$	Product rule	**Point 1 (P1)**
		Chain rule	**Point 2 (P2)**
	$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,y)=(1,-1)} = (3-1)(-1)^2 = 2$ $f''(1) = \left.\frac{d^2y}{dx^2}\right\|_{(x,y)=(1,-1)} = -(-1)^2 + (3-1)(2)(-1)(2) = -9$	$f''(1)$	**Point 3 (P3)**

Scoring Notes for Part A

- The expression $\frac{d^2y}{dx^2} = -y^2 + (3 - x)2y$ or $\frac{d^2y}{dx^2} = 6y - y^2 - 2xy$ earns **P1** but not **P2**. Such a response is not eligible for **P3**.
- The expression $\frac{d^2y}{dx^2} = -2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = 6y\left(\frac{dy}{dx}\right) - 2y\left(\frac{dy}{dx}\right)$ earns **P2** but not **P1**. Such a response is eligible for **P3** for a consistent answer of $f''(1) = 4$ or $f''(1) = -8$, respectively, which is found by correctly substituting correct values for x, y, and $\frac{dy}{dx}$.
- A response of $-(-1)^2 + (3-1)(2)(-1)(2)$ earns **P1**, **P2**, and **P3** regardless of any subsequent errors in simplification.
- Alternate approach (using separation of variables):
  The particular solution for the differential equation that passes through the point $(1, -1)$ is $y = \frac{2}{x^2 - 6x + 3}$. Therefore, $\frac{dy}{dx} = \frac{-2(2x-6)}{\left(x^2 - 6x + 3\right)^2}$.
  This response has not yet earned **P1**, **P2**, or **P3**.
  - A response that correctly applies the quotient rule (or product rule) and the chain rule to find that $\frac{d^2y}{dx^2} = \frac{-4\left(x^2 - 6x + 3\right)^2 + 4(2x-6)\left(x^2 - 6x + 3\right)(2x-6)}{\left(x^2 - 6x + 3\right)^4}$ earns **P1** and **P2** and is eligible to earn **P3** for the correct answer of $f''(1) = -9$.
  - A response that correctly applies the quotient rule (or product rule) but does not correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4\left(x^2 - 6x + 3\right)^2 + 4(2x-6)\left(x^2 - 6x + 3\right)}{\left(x^2 - 6x + 3\right)^4}$) earns **P1**, does not earn **P2**, and is not eligible to earn **P3**.
  - A response that does not correctly apply the quotient rule (or product rule) but does correctly apply the chain rule (e.g., $\frac{d^2y}{dx^2} = \frac{-4}{2\left(x^2 - 6x + 3\right)(2x-6)}$) does not earn **P1**, earns **P2**, and is eligible to earn **P3** for a consistent answer.

B Write the second-degree Taylor polynomial for f about $x = 1$.

$f'(1) = 2$ and $f''(1) = -9$	Two terms	**Point 4 (P4)**
$P_2(x) = -1 + 2(x-1) - \frac{9}{2}(x-1)^2$	Remaining term	**Point 5 (P5)**

Scoring Notes for Part B

- **P4** and **P5** can be earned with an answer consistent with incorrect values of $f'(1)$ and $f''(1)$ imported from part A.
- Any terms of degree greater than two or "$+\cdots$" does not earn **P5**.
- A response of $-1 + 2(x-1) - \frac{9}{2}(x-1)^2$ earns **P4** and **P5**, regardless of any subsequent algebraic simplification.
- A response that does not present the polynomial as powers of $(x-1)$ but instead presents a correct expanded/simplified form of the polynomial (e.g., $-\frac{9}{2}x^2 + 11x - \frac{15}{2}$) earns **P4** but not **P5**.

C The second-degree Taylor polynomial for f about $x = 1$ is used to approximate $f(1.1)$. Given that $|f'''(x)| \le 60$ for all x in the interval $1 \le x \le 1.1$, use the Lagrange error bound to show that this approximation differs from $f(1.1)$ by at most 0.01.

$\lvert f(1.1) - P_2(1.1)\rvert \le \frac{\max\limits_{1\le x\le 1.1}\lvert f'''(x)\rvert}{3!}\lvert 1.1-1\rvert^3 \le \frac{60}{6}(0.1)^3 = 0.01$	Form of error bound	**Point 6 (P6)**
	Analysis	**Point 7 (P7)**

Scoring Notes for Part C

- **P6** is earned for presenting either $\frac{\max\limits_{1\le x\le 1.1}|f'''(x)|}{3!}|1.1-1|^3$ or $\frac{60}{6}(0.1)^3$. Subsequent errors in simplification will not earn **P7**.
- To earn **P7**, a response must have earned **P6** and must explicitly connect the error bound with 0.01; for example by communicating $\text{Error} \le 0.01$, $\text{Error Bound} = 0.01$, or equivalent.
- A response that declares the error is equal to 0.01 (or any equivalent form of this value) does not earn **P7**.

D Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the work that leads to your answer.

$f(1.2) \approx f(1) + (1.2-1)\cdot \left.\frac{dy}{dx}\right\rvert_{(1,\,-1)}$ $= -1 + 0.2(2) = -0.6$	First step of Euler's method	**Point 8 (P8)**
$f(1.4) \approx f(1.2) + (1.4-1.2)\cdot \left.\frac{dy}{dx}\right\rvert_{(1.2,\,-0.6)}$ $\approx -0.6 + 0.2(3-1.2)(-0.6)^2 = -0.4704$ An approximation of $f(1.4)$ is -0.47.	Answer with supporting work	**Point 9 (P9)**

Scoring Notes for Part D

- To earn **P8**, a response must demonstrate the first step of Euler's method, with the correct initial condition, correct step size, and correct (or imported) expression for the derivative.

  Note: Any subsequent error in simplification or rounding will not affect the scoring for **P8**.
- The two steps of Euler's method may be explicit expressions or may be presented in a table. For example:

x	y	$\frac{dy}{dx}\cdot\Delta x$ (or $\frac{dy}{dx}\cdot 0.2$)
1	-1	0.4
1.2	-0.6	0.1296
1.4	-0.4704	

  Note: In the presence of a correct answer, a table does not need to be labeled to earn both **P8** and **P9**. In the presence of no answer or an incorrect answer, such a table must be correctly labeled to earn **P8**.
- A response of $-0.6 + 0.2(3-1.2)(-0.6)^2$ earns **P9**, regardless of any subsequent errors in simplification or rounding.
- A response that imports an incorrect value for $f'(1)$ from part A or part B is eligible to earn **P9** with a consistent answer.
- A response may report the final answer as -0.47, $(1.4, -0.47)$, $-\frac{294}{625}$, or equivalent.

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Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$	$2 : \frac{d^2y}{dx^2}$ in terms of x and y
(b) $\left.\frac{dy}{dx}\right\|_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$ $\left.\frac{d^2y}{dx^2}\right\|_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$ Thus, the graph of f has a relative maximum at the point $(-2, 8)$.	2 : conclusion with justification
(c) $\lim_{x\to -1}(g(x) - 2) = 0$ and $\lim_{x\to -1} 3(x+1)^2 = 0$ Using L'Hospital's Rule, $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right)$ $\lim_{x\to -1} g'(x) = 0$ and $\lim_{x\to -1} 6(x+1) = 0$ Using L'Hospital's Rule, $\lim_{x\to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x\to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$	$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$ $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$	$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$