Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

- The model solution is presented using standard mathematical notation.
- Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be accurate to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Taylor series for a function f about $x = 4$ is given by $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n} = \frac{(x-4)^2}{2 \cdot 3} + \frac{(x-4)^3}{3 \cdot 3^2} + \frac{(x-4)^4}{4 \cdot 3^3} + \cdots + \frac{(x-4)^{n+1}}{(n+1)3^n} + \cdots$ and converges to $f(x)$ on its interval of convergence.

	Model Solution	Scoring	
A	Using the ratio test, find the interval of convergence of the Taylor series for f about $x = 4$. Justify your answer.		
	$\left\lvert \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}} \right\rvert = \left\lvert \frac{(x-4)(n+1)}{3(n+2)} \right\rvert$	Sets up ratio	**Point 1 (P1)**
	$\lim_{n\to\infty} \left\lvert \frac{(x-4)(n+1)}{3(n+2)} \right\rvert = \left\lvert \frac{x-4}{3} \right\rvert \lim_{n\to\infty} \frac{n+1}{n+2} = \left\lvert \frac{x-4}{3} \right\rvert$	Limit of ratio	**Point 2 (P2)**
	$\left\lvert \frac{x-4}{3} \right\rvert < 1$ when $-3 < x-4 < 3$. Thus, the series converges when $1 < x < 7$.	Interior of interval of convergence	**Point 3 (P3)**
	When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3}{n+1}$, which converges by the alternating series test. When $x = 7$, the series is $\sum_{n=1}^{\infty} \frac{3}{n+1}$, which diverges by limit comparison to the harmonic series. Therefore, the series converges for $1 \le x < 7$.	Considers both endpoints	**Point 4 (P4)**
		Analysis and interval of convergence	**Point 5 (P5)**

Scoring Notes for Part A

- **P1** is earned by presenting a correct ratio with or without absolute values. Once earned, **P1** is banked (i.e., subsequent errors in simplification or evaluation do not impact scoring for **P1**). **P2** is not earned if there are any errors in simplification or evaluation of the limit.
- **P1** is earned for ratios mathematically equivalent to any of the following:

$$\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-4)^{n+1}},\ \frac{\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}}{\frac{(x-4)^{n+1}}{(n+1)3^n}},\ \frac{(x-4)^{n+1}}{(n+1)3^n}\cdot\frac{n3^{n-1}}{(x-4)^n},\text{ or }\frac{\frac{(x-4)^{n+1}}{(n+1)3^n}}{\frac{(x-4)^n}{n3^{n-1}}}.$$

- **P1** is also earned for ratios mathematically equivalent to the following reciprocal ratios:

$$\frac{(x-4)^{n+1}}{(n+1)3^n}\cdot\frac{(n+2)3^{n+1}}{(x-4)^{n+2}},\ \frac{\frac{(x-4)^{n+1}}{(n+1)3^n}}{\frac{(x-4)^{n+2}}{(n+2)3^{n+1}}},\ \frac{(x-4)^n}{n3^{n-1}}\cdot\frac{(n+1)3^n}{(x-4)^{n+1}},\text{ or }\frac{\frac{(x-4)^n}{n3^{n-1}}}{\frac{(x-4)^{n+1}}{(n+1)3^n}}.$$

  A response that presents any of these reciprocal ratios earns **P1** and is eligible for **P2**, **P3**, and **P4**, but not **P5**.
- A response that does not present a ratio is not eligible for **P2**.
- A response that does not use the absolute value of a ratio can earn both **P1** and **P2**.
- A response that does not use absolute value in computing the limit is eligible for **P3** if the expression $-1<\frac{x-4}{3}<1$ or an equivalent inequality is presented.
- A response that presents an incorrect limit of form $\frac{|x-4|}{b}$, where $b>0$, is eligible for **P3**.

  P3 is then earned for correctly finding an interval with interior $(1, 7)$ or $(4-b, 4+b)$.

  Note: $|x-4|<b$ is not sufficient to earn **P3**. $x-4<b$ can earn **P3** if this is resolved to $-b+4<x<b+4$.
- **P4** is earned for considering both endpoints of the correct interval or both endpoints of an incorrect interval that has earned **P3**.
- To earn **P5**, a response must correctly analyze the series at $x=1$ and $x=7$, and present the correct interval of convergence. Naming of an appropriate test is sufficient for the analysis at each endpoint. In addition to the tests listed in the model solution, the direct comparison test to an appropriate series or the integral test may also be used.

B Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 4$.

$f'(x) = \dfrac{(x-4)}{3} + \dfrac{(x-4)^2}{9} + \dfrac{(x-4)^3}{27} + \cdots + \dfrac{(x-4)^n}{3^n} + \cdots$	First three terms	**Point 6 (P6)**
	General term	**Point 7 (P7)**

Scoring Notes for Part B

- **P6** is earned by presenting the first three nonzero terms in a list or as part of a polynomial or series.
- **P7** is earned by identifying the correct general term (either individually or as part of a polynomial or series).

C The Taylor series for f' described in part B is a geometric series. For all x in the interval of convergence of the Taylor series for f', show that $f'(x) = \dfrac{x-4}{7-x}$.

The Taylor series for f' is a geometric series with first term $\dfrac{x-4}{3}$ and common ratio $\dfrac{x-4}{3}$. $f'(x) = \dfrac{\frac{x-4}{3}}{1-\frac{x-4}{3}} = \dfrac{x-4}{3-(x-4)} = \dfrac{x-4}{7-x}$	Verification	**Point 8 (P8)**

Scoring Notes for Part C

- A response of $f'(x) = \dfrac{\frac{x-4}{3}}{1-\frac{x-4}{3}}$ is sufficient to earn **P8**.

D	It is known that the radius of convergence of the Taylor series for f about $x = 4$ is the same as the radius of convergence of the Taylor series for f' about $x = 4$. Does the Taylor series for f' described in part B converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$? Give a reason for your answer.		
	It follows from the work in part A that the interior of the interval of convergence of the Taylor series for f' is $1 < x < 7$. Therefore, $x = 8$ would be outside the interval of convergence of f', and the Taylor series for f' would not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.	Answer with reason	**Point 9 (P9)**

Scoring Notes for Part D

- A response of "no, $x = 8$ is outside the interval of convergence" is sufficient to earn **P9**.
- **P9** can be earned with a response consistent with an incorrect interval of convergence imported from part A.
- Alternate solutions:
  - Because the series for $f'(x)$ is geometric, this converges to $f(x)$ for all values of x such that the common ratio $\frac{x-4}{3}$ is between -1 and 1.

    $-1 < \frac{x-4}{3} < 1 \Rightarrow -3 < x - 4 < 3 \Rightarrow 1 < x < 7$
  - Because $x = 8$ is outside the interval $1 < x < 7$, the series for f' does not converge to $f'(x) = \frac{x-4}{7-x}$ at $x = 8$.
  - For $x = 8$, the series is given by $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$, which is a geometric series with $r = \frac{4}{3} > 1$.

    Therefore, the series diverges for $x = 8$.