

9.9 Finding the Area of the Region Bounded by Two Polar Curves

Name: _____ Class: _____ Date: _____

Total: 13 marks

Objective

Build the skills to answer exam questions on the **area between two polar curves**.**You must be able to:**

- find the **intersection angles** of two polar curves ($r_1 = r_2$)
- integrate $\frac{1}{2}(r_{\text{outer}}^2 - r_{\text{inner}}^2)$
- decide which curve is outer over each interval

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ The formula

The area between an outer curve r_1 and an inner curve r_2 (over the same θ -range) is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta.$$

Subtract the inner sector from the outer sector.

■ Finding intersection angles

Set the curves equal. For $r = 3$ and $r = 2 + 2 \cos \theta$: $3 = 2 + 2 \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$.

■ Which is outer?

Test a θ between the intersections. Between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, at $\theta = 0$: $r = 3$ vs $r = 4$, so the cardioid $2 + 2 \cos \theta$ is outer there.

■ Setting up

$$A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} ((2 + 2 \cos \theta)^2 - 3^2) d\theta.$$

Use symmetry to write it as $2 \times \frac{1}{2} \int_0^{\pi/3} (\dots) d\theta$ if convenient.

2 Practice

Now apply the methods above.

2.1 State the area-between-two-polar-curves formula. [1]

2.2 Find the intersection angles of $r = 2$ and $r = 4 \cos \theta$. [3]2.3 Between $\theta = 0$ and $\theta = \frac{\pi}{6}$, which is larger: $r = 2$ or $r = 4 \cos \theta$? [1]

3 Exam-style questions

3.1 The area between an outer curve r_1 and inner r_2 is [1]

- A $\frac{1}{2} \int (r_1 - r_2)^2 d\theta$
- B $\frac{1}{2} \int (r_1^2 - r_2^2) d\theta$
- C $\int (r_1^2 - r_2^2) d\theta$
- D $\frac{1}{2} \int (r_1^2 + r_2^2) d\theta$

3.2 Two curves are $r = 1$ (inner) and $r = 2 \cos \theta$ (outer) where they overlap.

(a) Find the intersection angles. [2]

(b) Set up the area inside $r = 2 \cos \theta$ and outside $r = 1$. [3]

3.3 Explain why you must square the radii **before** subtracting, i.e. why $r_1^2 - r_2^2 \neq (r_1 - r_2)^2$ in the area formula. ^[2]

Solutions

2.1 $A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta.$

2.2 $2 = 4 \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}.$

2.3 $r = 4 \cos \theta$ is larger (at $\theta = 0$: $4 > 2$).

3.1 B — half the integral of the difference of the squared radii.

3.2 (a) $1 = 2 \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}.$ (b) $A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} ((2 \cos \theta)^2 - 1^2) d\theta.$

3.3 Each sector's area is $\frac{1}{2} r^2 d\theta$, so the region between is $\frac{1}{2} (r_1^2 - r_2^2) d\theta$ — a difference of the two **squared** radii; $(r_1 - r_2)^2$ would wrongly square the difference of the radii instead.