AP CALCULUS BC • EXERCISE SHEET

9.9 Finding the Area of the Region Bounded by Two Polar Curves

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **area between two polar curves**.

You must be able to:

- find the **intersection angles** of two polar curves ($r_1 = r_2$)
- integrate $\frac{1}{2}\left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right)$
- decide which curve is outer over each interval

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ The formula

The area between an outer curve r_1 and an inner curve r_2 (over the same θ-range) is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right) d\theta.$$

Subtract the inner sector from the outer sector.

■ Finding intersection angles

Set the curves equal. For $r = 3$ and $r = 2 + 2\cos\theta$: $3 = 2 + 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$.

■ Which is outer?

Test a θ between the intersections. Between $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, at $\theta = 0$: $r = 3$ vs $r = 4$, so the cardioid $2 + 2\cos\theta$ is outer there.

■ Setting up

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((2 + 2\cos\theta)^2 - 3^2\right) d\theta.$$

Use symmetry to write it as $2 \times \frac{1}{2}\int_0^{\pi/3}(\dots)\, d\theta$ if convenient.

2 Practice

Now apply the methods above.

2.1 State the area-between-two-polar-curves formula. [1]

2.2 Find the intersection angles of $r = 2$ and $r = 4\cos\theta$. [3]

2.3 Between $\theta = 0$ and $\theta = \frac{\pi}{6}$, which is larger: $r = 2$ or $r = 4\cos\theta$? [1]

3 Exam-style questions

3.1 The area between an outer curve r_1 and inner r_2 is [1]

- **A** $\frac{1}{2}\int (r_1 - r_2)^2 \, d\theta$
- **B** $\frac{1}{2}\int (r_1^2 - r_2^2) \, d\theta$
- **C** $\int (r_1^2 - r_2^2) \, d\theta$
- **D** $\frac{1}{2}\int (r_1^2 + r_2^2) \, d\theta$

3.2 Two curves are $r = 1$ (inner) and $r = 2\cos\theta$ (outer) where they overlap.

(a) Find the intersection angles. [2]

(b) Set up the area inside $r = 2\cos\theta$ and outside $r = 1$. [3]

3.3 Explain why you must square the radii **before** subtracting, i.e. why $r_1^2 - r_2^2 \neq (r_1 - r_2)^2$

in the area formula. [2]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.9 Finding the Area of the Region Bounded by Two Polar Curves** lesson on the **Learn** page;
- read the **Finding the Area of the Region Bounded by Two Polar Curves** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right) d\theta.$

2.2 $2 = 4\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}.$

2.3 $r = 4\cos\theta$ is larger (at $\theta = 0$: $4 > 2$).

3.1 **B** —half the integral of the difference of the squared radii.

3.2 (a) $1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$. (b) $A = \dfrac{1}{2}\displaystyle\int_{-\pi/3}^{\pi/3} \left((2\cos\theta)^2 - 1^2\right) d\theta.$

3.3 Each sector's area is $\frac{1}{2}r^2\,d\theta$, so the region between is $\frac{1}{2}(r_1^2 - r_2^2)\,d\theta$ —a difference of the two **squared** radii; $(r_1 - r_2)^2$ would wrongly square the difference of the radii instead.