9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on **planar motion** with parametric and vector functions.

You must be able to:

- find velocity, acceleration, and **speed** from a position vector
- find the **position at a later time** (start $+ \int \vec{v}\,dt$)
- find the **total distance travelled** $\int_a^b \sqrt{x'^2 + y'^2}\,dt$

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ The motion toolkit

For a particle at $\vec{r}(t) = \langle x(t), y(t)\rangle$: velocity $\vec{v} = \langle x', y'\rangle$, acceleration $\vec{a} = \langle x'', y''\rangle$, and speed $|\vec{v}| = \sqrt{x'^2 + y'^2}$.

■ Position at a later time

If you know $\vec{r}(a)$ and $\vec{v}(t)$, the position at b is

$$\vec{r}(b) = \vec{r}(a) + \int_a^b \vec{v}(t)\,dt,$$

component by component.

■ Total distance travelled

The **distance travelled** (not displacement) over $[a, b]$ is

$$\int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\,dt$$

—the parametric arc length of the path.

■ **A worked speed**

For $\vec{v}(t) = \langle 3t, 4t \rangle$: speed $= \sqrt{9t^2 + 16t^2} = 5t$; at $t = 2$, speed $= 10$.

2 Practice

Now apply the methods above.

2.1 For $\vec{v}(t) = \langle 6, 8 \rangle$, find the speed. [1]

2.2 A particle has $\vec{v}(t) = \langle 2t, 2 \rangle$. Set up the total distance travelled over $0 \le t \le 3$. [2]

2.3 A particle starts at $\vec{r}(0) = \langle 1, 2 \rangle$ with $\vec{v}(t) = \langle 4, 6t \rangle$. Find $\vec{r}(t)$. [3]

3 Exam-style questions

3.1 The total distance a particle travels over $[a, b]$ is [1]

- **A** $\displaystyle\int_a^b \vec{v}\,dt$
- **B** $|\vec{r}(b) - \vec{r}(a)|$
- **C** $\displaystyle\int_a^b \sqrt{x'^2 + y'^2}\,dt$
- **D** $\vec{r}(b) - \vec{r}(a)$

3.2 A particle moves with $\vec{v}(t) = \langle 3t^2, 2t \rangle$, starting at $\vec{r}(0) = \langle 0, 1 \rangle$.

(a) Find the position $\vec{r}(t)$. [3]

(b) Find the speed at $t = 1$. [2]

3.3 A particle has $\vec{v}(t) = \langle \cos t, \sin t \rangle$. Set up and evaluate the total distance travelled over $0 \leq t \leq \pi$. [3]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions** lesson on the **Learn** page;
- read the **Solving Motion Problems Using Parametric and Vector-Valued Functions** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\sqrt{36+64} = 10$.

2.2 $\displaystyle\int_0^3 \sqrt{(2t)^2 + 2^2}\,dt = \int_0^3 \sqrt{4t^2+4}\,dt$.

2.3 $\vec{r}(t) = \langle 4t+1, 3t^2+2\rangle$.

3.1 **C** —total distance is the arc-length integral of speed.

3.2 (a) $\vec{r}(t) = \langle t^3, t^2+1\rangle$. (b) $\vec{v}(1) = \langle 3, 2\rangle$; speed $= \sqrt{9+4} = \sqrt{13}$.

3.3 speed $= \sqrt{\cos^2 t + \sin^2 t} = 1$; distance $= \displaystyle\int_0^\pi 1\,dt = \pi$.