AP CALCULUS BC • EXERCISE SHEET

9.5 Integrating Vector-Valued Functions

Name: ____________ Class: ________ Date: __________ **Total: 16 marks**

KEY WORDS vector-valued function 向量值函数

Objective

Build the skills to answer exam questions on **integrating vector-valued functions**.

You must be able to:

- integrate a vector function **component by component**
- use an **initial condition** to find each constant
- recover velocity from acceleration, or position from velocity

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Integrate each component

$\int \vec{r}(t)\,dt$ integrates each component separately, giving a **vector** of constants $\vec{C}$:

$$\int \langle f(t), g(t)\rangle\,dt = \left\langle \int f\,dt, \int g\,dt \right\rangle + \vec{C}.$$

From acceleration to velocity

Given $\vec{a}(t) = \langle 2, -6t\rangle$ and $\vec{v}(0) = \langle 1, 0\rangle$: integrate, $\vec{v}(t) = \langle 2t + C_1, -3t^2 + C_2\rangle$. Apply $\vec{v}(0) = \langle 1, 0\rangle$: $C_1 = 1$, $C_2 = 0$, so $\vec{v}(t) = \langle 2t + 1, -3t^2\rangle$.

From velocity to position

Integrate $\vec{v}$ and use $\vec{r}(0)$ to fix the constants. Each component is a separate initial-value problem.

Definite integral of a vector

$\int_a^b \vec{v}(t)\,dt$ gives the **displacement** vector over $[a, b]$ — integrate each component between the limits.

2 Practice

Now apply the methods above.

2.1 Find $\int \langle 2t, 3\rangle\,dt$ (include the constant vector). [2]

2.2 Given $\vec{a}(t) = \langle 0, -10\rangle$ and $\vec{v}(0) = \langle 5, 0\rangle$, find $\vec{v}(t)$. [3]

2.3 Evaluate $\int_0^2 \langle 1, 2t\rangle\,dt$. [2]

3 Exam-style questions

3.1 To integrate a vector-valued function you [1]

- **A** integrate each component separately
- **B** take the magnitude first
- **C** differentiate it
- **D** add the components then integrate

3.2 A particle has acceleration $\vec{a}(t) = \langle 6t, 2 \rangle$, with $\vec{v}(0) = \langle 0, 1 \rangle$.

(a) Find $\vec{v}(t)$. [3]

(b) Find $\vec{v}(1)$. [1]

3.3 A particle has velocity $\vec{v}(t) = \langle 2t, \cos t \rangle$ and starts at $\vec{r}(0) = \langle 1, 0 \rangle$. Find the position $\vec{r}(t)$. [4]

Solutions

2.1 $\langle t^2,\, 3t \rangle + \vec{C}$.

2.2 $\vec{v}(t) = \langle C_1, -10t + C_2 \rangle$; $\vec{v}(0) = \langle 5, 0 \rangle \Rightarrow C_1 = 5, C_2 = 0$; $\vec{v}(t) = \langle 5, -10t \rangle$.

2.3 $\left\langle [t]_0^2, [t^2]_0^2 \right\rangle = \langle 2, 4 \rangle$.

3.1 **A** — integrate each component separately.

3.2 (a) $\vec{v}(t) = \langle 3t^2 + C_1, 2t + C_2 \rangle$; $\vec{v}(0) = \langle 0, 1 \rangle \Rightarrow C_1 = 0, C_2 = 1$; $\vec{v}(t) = \langle 3t^2, 2t + 1 \rangle$. (b) $\vec{v}(1) = \langle 3, 3 \rangle$.

3.3 $\vec{r}(t) = \langle t^2 + C_1, \sin t + C_2 \rangle$; $\vec{r}(0) = \langle 1, 0 \rangle \Rightarrow C_1 = 1, C_2 = 0$; $\vec{r}(t) = \langle t^2 + 1, \sin t \rangle$.