9.5 Integrating Vector-Valued Functions

Name: ______________ Class: __________ Date: ____________ **Total: 16 marks**

Objective

Build the skills to answer exam questions on **integrating vector-valued functions**.

You must be able to:

- integrate a vector function **component by component**
- use an **initial condition** to find each constant
- recover velocity from acceleration, or position from velocity

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ Integrate each component

$\int \vec{r}(t)\,dt$ integrates each component separately, giving a **vector** of constants $\vec{C}$:

$$\int \langle f(t), g(t) \rangle\,dt = \left\langle \int f\,dt, \int g\,dt \right\rangle + \vec{C}.$$

■ From acceleration to velocity

Given $\vec{a}(t) = \langle 2, -6t \rangle$ and $\vec{v}(0) = \langle 1, 0 \rangle$: integrate, $\vec{v}(t) = \langle 2t + C_1, -3t^2 + C_2 \rangle$. Apply $\vec{v}(0) = \langle 1, 0 \rangle$: $C_1 = 1$, $C_2 = 0$, so $\vec{v}(t) = \langle 2t + 1, -3t^2 \rangle$.

■ From velocity to position

Integrate $\vec{v}$ and use $\vec{r}(0)$ to fix the constants. Each component is a separate initial-value problem.

■ Definite integral of a vector

$\int_a^b \vec{v}(t)\,dt$ gives the **displacement** vector over $[a, b]$ —integrate each component between the limits.

2 Practice

Now apply the methods above.

2.1 Find $\int \langle 2t,\, 3\rangle \, dt$ (include the constant vector). [2]

2.2 Given $\vec{a}(t) = \langle 0, -10\rangle$ and $\vec{v}(0) = \langle 5, 0\rangle$, find $\vec{v}(t)$. [3]

2.3 Evaluate $\int_0^2 \langle 1,\, 2t\rangle \, dt$. [2]

3 Exam-style questions

3.1 To integrate a vector-valued function you [1]

- **A** integrate each component separately
- **B** take the magnitude first
- **C** differentiate it
- **D** add the components then integrate

3.2 A particle has acceleration $\vec{a}(t) = \langle 6t, 2\rangle$, with $\vec{v}(0) = \langle 0, 1\rangle$.

(a) Find $\vec{v}(t)$. [3]

(b) Find $\vec{v}(1)$. [1]

3.3 A particle has velocity $\vec{v}(t) = \langle 2t, \cos t \rangle$ and starts at $\vec{r}(0) = \langle 1, 0 \rangle$. Find the position $\vec{r}(t)$. [4]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.5 Integrating Vector-Valued Functions** lesson on the **Learn** page;
- read the **Integrating Vector-Valued Functions** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\langle t^2,\, 3t\rangle + \vec{C}$.

2.2 $\vec{v}(t) = \langle C_1, -10t + C_2\rangle$; $\vec{v}(0) = \langle 5, 0\rangle \Rightarrow C_1 = 5, C_2 = 0$; $\vec{v}(t) = \langle 5, -10t\rangle$.

2.3 $\left\langle [t]_0^2, [t^2]_0^2 \right\rangle = \langle 2, 4\rangle$.

3.1 **A** —integrate each component separately.

3.2 (a) $\vec{v}(t) = \langle 3t^2 + C_1, 2t + C_2\rangle$; $\vec{v}(0) = \langle 0, 1\rangle \Rightarrow C_1 = 0, C_2 = 1$; $\vec{v}(t) = \langle 3t^2, 2t + 1\rangle$. (b) $\vec{v}(1) = \langle 3, 3\rangle$.

3.3 $\vec{r}(t) = \langle t^2 + C_1, \sin t + C_2\rangle$; $\vec{r}(0) = \langle 1, 0\rangle \Rightarrow C_1 = 1, C_2 = 0$; $\vec{r}(t) = \langle t^2 + 1, \sin t\rangle$.