9.4 Defining and Differentiating Vector-Valued Functions

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on **vector-valued functions** —differentiating a position vector.

You must be able to:

- read a **vector-valued function 向量值函数** $\vec{r}(t) = \langle x(t), y(t) \rangle$
- differentiate **component-wise** to get velocity and acceleration
- find the **speed** $|\vec{v}| = \sqrt{x'^2 + y'^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ Position, velocity, acceleration

A vector function gives a position for each t: $\vec{r}(t) = \langle x(t), y(t) \rangle$. Differentiate each component:

$$\vec{v}(t) = \langle x'(t), y'(t) \rangle, \qquad \vec{a}(t) = \langle x''(t), y''(t) \rangle.$$

■ A worked derivative

For $\vec{r}(t) = \langle t^2, \sin t \rangle$: $\vec{v}(t) = \langle 2t, \cos t \rangle$ and $\vec{a}(t) = \langle 2, -\sin t \rangle$.

■ Speed is a magnitude

The **speed** is the length of the velocity vector:

$$|\vec{v}| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}.$$

For $\vec{v} = \langle 3, 4 \rangle$, speed $= \sqrt{9 + 16} = 5$.

■ Evaluating at a point

At a specific t, plug in to get the numeric velocity, acceleration, and speed. At $t = 0$ for $\vec{r} = \langle t^2, \sin t \rangle$: $\vec{v}(0) = \langle 0, 1 \rangle$, speed $= 1$.

2 Practice

Now apply the methods above.

2.1 For $\vec{r}(t) = \langle t^3, t^2 \rangle$, find $\vec{v}(t)$. [2]

2.2 For $\vec{v}(t) = \langle 2t, 3 \rangle$, find the speed at $t = 2$. [2]

2.3 For $\vec{r}(t) = \langle \cos t, \sin t \rangle$, find $\vec{a}(t)$. [2]

3 Exam-style questions

3.1 The speed of a particle with velocity $\vec{v} = \langle x', y' \rangle$ is [1]

- **A** $x' + y'$
- **B** $\sqrt{x'^2 + y'^2}$
- **C** $\langle x', y' \rangle$
- **D** $x'y'$

3.2 A particle has position $\vec{r}(t) = \langle t^2 - 1,\ 2t \rangle$.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$. [2]

(b) Find the speed at $t = 1$. [2]

3.3 For $\vec{r}(t) = \langle e^t, e^{-t} \rangle$, find the velocity and speed at $t = 0$. [4]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.4 Defining and Differentiating Vector-Valued Functions** lesson on the **Learn** page;
- read the **Defining and Differentiating Vector-Valued Functions** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\vec{v}(t) = \langle 3t^2, 2t \rangle$.

2.2 $\vec{v}(2) = \langle 4, 3 \rangle$; speed $= \sqrt{16 + 9} = 5$.

2.3 $\vec{v} = \langle -\sin t, \cos t \rangle$; $\vec{a} = \langle -\cos t, -\sin t \rangle$.

3.1 **B** —speed is the magnitude $\sqrt{x'^2 + y'^2}$.

3.2 (a) $\vec{v} = \langle 2t, 2 \rangle$, $\vec{a} = \langle 2, 0 \rangle$. (b) $\vec{v}(1) = \langle 2, 2 \rangle$; speed $= \sqrt{4 + 4} = 2\sqrt{2}$.

3.3 $\vec{v} = \langle e^t, -e^{-t} \rangle$; at $t = 0$, $\vec{v} = \langle 1, -1 \rangle$; speed $= \sqrt{1 + 1} = \sqrt{2}$.