AP CALCULUS BC • EXERCISE SHEET

9.3 Finding Arc Lengths of Curves Given by Parametric Equations

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **arc length of a parametric curve**.

You must be able to:

- apply $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- set up the integral from parametric equations
- recognise it as summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

The parametric arc-length formula

For $x = x(t)$, $y = y(t)$ with t from a to b:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Setting it up

For $x = t^2$, $y = t^3$ on $[0, 1]$: $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$, so

$$L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt.$$

A curve that simplifies

For a circle $x = \cos t$, $y = \sin t$: $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, so the integrand is $\sqrt{\sin^2 t + \cos^2 t} = 1$. Then $L = \int_0^{2\pi} 1\, dt = 2\pi$ (the circumference).

Same idea as before

This is the $\sqrt{dx^2 + dy^2}$ hypotenuse summed along the curve, now with both x and y depending on t.

2 Practice

Now apply the methods above.

2.1 State the parametric arc-length formula. [1]

2.2 For $x = t^2$, $y = t^3$, write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$. [2]

2.3 For $x = 3t$, $y = 4t$ on $[0, 2]$, set up and evaluate the arc length. [3]

3 Exam-style questions

3.1 The parametric arc length uses the integrand [1]

- **A** $\sqrt{1 + (dy/dx)^2}$
- **B** $\sqrt{(dx/dt)^2 + (dy/dt)^2}$
- **C** $\dfrac{dy/dt}{dx/dt}$
- **D** $(dx/dt)(dy/dt)$

3.2 A curve is $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$.

(a) Show the integrand simplifies to 1. [2]

(b) Evaluate the arc length. [1]

3.3 Set up (do not evaluate) the arc-length integral for $x = t - \sin t$, $y = 1 - \cos t$ on

$0 \leq t \leq 2\pi$. [3]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.3 Finding Arc Lengths of Curves Given by Parametric Equations** lesson on the **Learn** page;
- read the **Finding Arc Lengths of Curves Given by Parametric Equations** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $L = \displaystyle\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$

2.2 $(2t)^2 + (3t^2)^2 = 4t^2 + 9t^4.$

2.3 $\sqrt{9+16} = 5$; $L = \displaystyle\int_0^2 5\, dt = 10.$

3.1 **B** —the parametric integrand is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.

3.2 (a) $(-\sin t)^2 + (\cos t)^2 = \sin^2 t + \cos^2 t = 1$. (b) $\int_0^{\pi} 1\, dt = \pi$.

3.3 $\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$; $L = \displaystyle\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt.$