9.2 Second Derivatives of Parametric Equations

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

Objective

Build the skills to answer exam questions on the **second derivative of a parametric curve** —testing concavity.

You must be able to:

- compute $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$
- avoid the wrong "divide the second derivatives" shortcut
- use the sign of $\dfrac{d^2y}{dx^2}$ to state **concavity** 凹凸性

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ The correct formula

The second derivative is **not** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t, then divide by $\dfrac{dx}{dt}$ again:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

■ A worked example

For $x = t^2$, $y = t^3$: $\dfrac{dy}{dx} = \dfrac{3t}{2}$. Then $\dfrac{d}{dt}\left(\dfrac{3t}{2}\right) = \dfrac{3}{2}$, and $\dfrac{dx}{dt} = 2t$, so

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}.$$

■ Concavity

At $t = 1$, $\dfrac{d^2y}{dx^2} = \frac{3}{4} > 0$: the curve is **concave up** there. A negative value means concave down.

■ **The two-step routine**

1. Find $\dfrac{dy}{dx}$ (a function of t). 2. Differentiate it w.r.t. t and divide by $\dfrac{dx}{dt}$.

2 Practice

Now apply the methods above.

2.1 State the formula for $\dfrac{d^2y}{dx^2}$ for a parametric curve. [1]

2.2 For $x = t^2$, $y = t^3$, given $\dfrac{dy}{dx} = \dfrac{3t}{2}$, find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$. [1]

2.3 Complete: find $\dfrac{d^2y}{dx^2}$ for that curve. [2]

3 Exam-style questions

3.1 The second derivative of a parametric curve is [1]

- **A** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$
- **B** $\dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$
- **C** $\dfrac{dy/dt}{dx/dt}$
- **D** $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$

3.2 A curve has $x = t$, $y = t^3 - 3t$.

(a) Find $\dfrac{dy}{dx}$. [1]

(b) Find $\dfrac{d^2y}{dx^2}$. [2]

(c) State the concavity at $t = 1$. [1]

3.3 For $x = t^2$, $y = t^2 + t$, find $\dfrac{d^2y}{dx^2}$ at $t = 1$. [4]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **9.2 Second Derivatives of Parametric Equations** lesson on the **Learn** page;
- read the **Second Derivatives of Parametric Equations** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}(dy/dx)}{dx/dt}$.

2.2 $\dfrac{d}{dt}\left(\dfrac{3t}{2}\right) = \dfrac{3}{2}$.

2.3 $\dfrac{d^2y}{dx^2} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$.

3.1 **B** —differentiate dy/dx w.r.t. t, then divide by dx/dt.

3.2 (a) $\dfrac{dy}{dx} = \dfrac{3t^2-3}{1} = 3t^2 - 3$. (b) $\dfrac{d}{dt}(3t^2-3) = 6t$; divide by $dx/dt = 1$: $\dfrac{d^2y}{dx^2} = 6t$. (c) at $t = 1$, $6 > 0$: concave up.

3.3 $\dfrac{dy}{dx} = \dfrac{2t+1}{2t}$; $\dfrac{d}{dt}\left(\dfrac{2t+1}{2t}\right) = \dfrac{d}{dt}\left(1 + \frac{1}{2t}\right) = -\dfrac{1}{2t^2}$; divide by $2t$: $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4t^3}$; at $t = 1$: $-\frac{1}{4}$.