AP CALCULUS BC • EXERCISE SHEET

8.8 Volumes with Cross Sections: Triangles and Semicircles

Name: ____________ Class: _________ Date: ___________ **Total: 13 marks**

KEY WORDS equilateral triangle 等边三角形 • semicircular 半圆 • cross sections 横截面

Objective

Build the skills to answer exam questions on **volumes by cross sections (triangles and semicircles)** — using the right area formula for the slice.

You must be able to:

- use $A = \frac{\sqrt{3}}{4}s^2$ for an **equilateral triangle** 等边三角形 slice
- use $A = \frac{\pi}{8}s^2$ for a **semicircular** 半圆 slice (diameter $= s$)
- substitute the area formula and integrate

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Equilateral-triangle cross sections

A slice that is an equilateral triangle with side s has area $A = \frac{\sqrt{3}}{4}s^2$. If $s = f(x)-g(x)$, then

$$V = \int_a^b \frac{\sqrt{3}}{4}\big(f(x) - g(x)\big)^2\,dx.$$

■ Semicircular cross sections

A semicircle with **diameter** s has radius $\frac{s}{2}$ and area $\frac{1}{2}\pi\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}s^2$. So

$$V = \int_a^b \frac{\pi}{8}\big(f(x) - g(x)\big)^2\,dx.$$

■ The method is always the same

Whatever the slice's shape, write its **area** in terms of the gap s, then integrate. Only the constant in front changes: 1 (square), $\frac{\sqrt{3}}{4}$ (triangle), $\frac{\pi}{8}$ (semicircle).

■ A worked triangle-section volume

Base bounded by $y = \sqrt{x}$ and the x-axis on $[0,4]$, equilateral-triangle sections: $s = \sqrt{x}$, so $V = \int_0^4 \frac{\sqrt{3}}{4}x\,dx = \frac{\sqrt{3}}{4}\cdot 8 = 2\sqrt{3}$.

2 Practice

Now apply the methods above.

2.1 State the area of an equilateral triangle with side s. [1]

2.2 State the area of a semicircle with diameter s. [1]

2.3 A solid has equilateral-triangle cross sections with side $s = x$ on $[0,2]$. Find its volume. [3]

3 Exam-style questions

3.1 A semicircular cross section with diameter s has area [1]

- **A** $\frac{\pi}{2}s^2$
- **B** $\frac{\pi}{4}s^2$
- **C** $\frac{\pi}{8}s^2$
- **D** πs^2

3.2 The base of a solid is bounded by $y = 2 - x$ and the axes in the first quadrant. Cross sections perpendicular to the x-axis are equilateral triangles.

(a) Write the side length s as a function of x. [1]

(b) Set up the volume integral. [2]

3.3 The base is the region between $y = \sqrt{x}$ and the x-axis on $[0, 4]$. Cross sections perpendicular to the x-axis are semicircles with diameter across the region. Set up and evaluate the volume. [4]

Solutions

2.1 $A = \frac{\sqrt{3}}{4}s^2$.

2.2 $A = \frac{\pi}{8}s^2$.

2.3 $V = \int_0^2 \frac{\sqrt{3}}{4}x^2\,dx = \frac{\sqrt{3}}{4}\left[\frac{x^3}{3}\right]_0^2 = \frac{\sqrt{3}}{4} \cdot \frac{8}{3} = \frac{2\sqrt{3}}{3}$.

3.1 C — radius $s/2$ gives $\frac{1}{2}\pi(s/2)^2 = \frac{\pi}{8}s^2$.

3.2 (a) $s = 2 - x$. (b) $\int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx$.

3.3 $s = \sqrt{x}$; $V = \int_0^4 \frac{\pi}{8}(\sqrt{x})^2\,dx = \frac{\pi}{8}\int_0^4 x\,dx = \frac{\pi}{8}\left[\frac{x^2}{2}\right]_0^4 = \frac{\pi}{8} \cdot 8 = \pi$.