8.7 Volumes with Cross Sections: Squares and Rectangles

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on **volumes by cross sections (squares and rectangles)** —integrating a slice's area.

You must be able to:

- write the **side length** of a slice as the gap between two curves
- integrate the **cross-sectional area** 横截面积: $V = \int_a^b A(x)\,dx$
- handle a square ($A = s^2$) or a rectangle cross section

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ Volume as an integral of area

A solid whose cross section perpendicular to the x-axis has area $A(x)$ has volume

$$V = \int_a^b A(x)\,dx.$$

”Integrate the cross-sectional area” is the whole method.

■ Square cross sections

If each cross section is a **square** with side equal to the gap between $y = f(x)$ (top) and $y = g(x)$ (bottom), then $s = f(x) - g(x)$ and

$$V = \int_a^b \big(f(x) - g(x)\big)^2\,dx.$$

■ A worked square-section volume

Base bounded by $y = \sqrt{x}$ and the x-axis on $[0, 4]$, square cross sections: $s = \sqrt{x}$, so

$$V = \int_0^4 (\sqrt{x})^2\,dx = \int_0^4 x\,dx = 8.$$

■ **Rectangle cross sections**

A rectangle of height $= k\times$ (base) has area $A = k\,s^2$ if height is proportional to the side; otherwise use the stated width and height. Always write $A(x)$ first, then integrate.

2 Practice

Now apply the methods above.

2.1 A slice has area $A(x) = x^2$. Write the volume integral over $[0, 3]$. [1]

2.2 A solid has square cross sections with side $s = 2x$ on $[0, 2]$. Find its volume. [3]

2.3 Evaluate $\int_0^1 (1-x)^2\,dx$. [2]

3 Exam-style questions

3.1 The volume of a solid with cross-sectional area $A(x)$ is [1]

- **A** $\int_a^b \sqrt{A(x)}\,dx$
- **B** $\int_a^b A(x)\,dx$
- **C** $\pi \int_a^b A(x)\,dx$
- **D** $\int_a^b A(x)^2\,dx$

3.2 The base of a solid is the region between $y = 4 - x^2$ and the x-axis. Cross sections perpendicular to the x-axis are squares.

(a) Write the side length s as a function of x. [1]

(b) Set up the volume integral (with limits). [2]

3.3 The base of a solid is bounded by $y = x$ and $y = x^2$. Cross sections perpendicular to the x-axis are squares with side equal to the gap between the curves. Set up and evaluate the volume. [5]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **8.7 Volumes by Cross Sections: Squares and Rectangles** lesson on the **Learn** page;
- read the **Volumes with Cross Sections: Squares and Rectangles** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $\int_0^3 x^2\,dx$.

2.2 $A = (2x)^2 = 4x^2$; $\int_0^2 4x^2\,dx = \left[\frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$.

2.3 $\int_0^1 (1-x)^2\,dx = \left[-\frac{(1-x)^3}{3}\right]_0^1 = 0 - (-\frac{1}{3}) = \frac{1}{3}$.

3.1 B —integrate the cross-sectional area.

3.2 (a) $s = 4 - x^2$. (b) $\int_{-2}^2 (4-x^2)^2\,dx$.

3.3 $s = x - x^2$ on $[0,1]$; $V = \int_0^1 (x-x^2)^2\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1$
$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$.