AP CALCULUS BC • EXERCISE SHEET

8.5 Finding the Area Between Curves Expressed as Functions of y

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

Objective

Build the skills to answer exam questions on the **area between curves integrating in** y — when the curves are functions of y.

You must be able to:

- integrate **right minus left**: $\int_c^d \big(x_{\text{right}} - x_{\text{left}}\big)\, dy$
- rewrite curves as $x = f(y)$
- choose dy to avoid splitting a region

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Right minus left

When a region is bounded left and right by curves, integrate in y:

$$\int_c^d \big(x_{\text{right}}(y) - x_{\text{left}}(y)\big)\, dy,$$

where c and d are the lower and upper y-limits.

■ Rewriting as x of y

$y = \sqrt{x}$ becomes $x = y^2$; $y = x$ stays $x = y$. For the region between $y = \sqrt{x}$ and $y = x$, in y: right curve $x = y$? Test $y = 0.5$: $y^2 = 0.25$ (left), $y = 0.5$ (right), so right $= y$, left $= y^2$.

■ Setting up the integral

Crossings of $x = y$ and $x = y^2$: $y = y^2 \Rightarrow y = 0, 1$. Area

$$\int_0^1 (y - y^2)\, dy = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{6}.$$

■ Why choose dy

Integrating in y turns a sideways-opening region (like one bounded by $x = y^2$) into a **single** integral, instead of splitting a top-and-bottom setup.

2 Practice

Now apply the methods above.

2.1 Rewrite $y = \sqrt{x}$ as a function $x = f(y)$. [1]

2.2 Find where $x = y$ meets $x = y^2$. [1]

2.3 Evaluate $\displaystyle\int_0^2 (4 - y^2)\, dy$. [2]

3 Exam-style questions

3.1 To find an area by integrating in y, you integrate [1]

- **A** top minus bottom
- **B** right minus left
- **C** $f(x) - g(x)$
- **D** the average of the curves

3.2 A region is bounded by $x = y^2$ and $x = 2y$.

(a) Find the y-limits of integration. [2]

(b) Set up and evaluate the area, integrating in y. [4]

3.3 Find the area enclosed by $x = y^2 - 1$ and $x = 3$ (integrate in y). [4]

Solutions

2.1 $x = y^2$ (for $y \geq 0$).

2.2 $y = y^2 \Rightarrow y = 0, 1$.

2.3 $\left[4y - \frac{y^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$.

3.1 B — integrating in y uses right minus left.

3.2 (a) $y^2 = 2y \Rightarrow y = 0, 2$. (b) right $= 2y$, left $= y^2$; $\int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.

3.3 $y^2 - 1 = 3 \Rightarrow y = \pm 2$; right $= 3$, left $= y^2 - 1$; $\int_{-2}^{2} \big(3 - (y^2 - 1)\big)\,dy = \int_{-2}^{2} (4 - y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = \frac{32}{3}$.