AP CALCULUS BC • EXERCISE SHEET

8.4 Finding the Area Between Curves Expressed as Functions of x

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

Objective

Build the skills to answer exam questions on the **area between two curves** (integrating in x).

You must be able to:

- find the **intersection points** that give the limits of integration
- integrate **top minus bottom**: $\int_a^b \big(f_{\text{top}} - f_{\text{bottom}}\big)\,dx$
- set up the area correctly even when a curve dips below the other

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ Top minus bottom

The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) from a to b is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

■ Finding the limits

Set the curves equal to find where they cross. For $y = x$ and $y = x^2$: $x = x^2 \Rightarrow x = 0, 1$. These are the limits.

■ Which is on top?

Test a point between the crossings. At $x = 0.5$: the line $y = 0.5$ is above the parabola $y = 0.25$, so $y = x$ is the **top**:

$$\int_0^1 (x - x^2)\,dx = \left[\tfrac{x^2}{2} - \tfrac{x^3}{3}\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}.$$

■

Area between $y = 4 - x^2$ and the x-axis ($y = 0$): crossings at $x = \pm 2$, so $\displaystyle\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \tfrac{x^3}{3}\right]_{-2}^{2} = \tfrac{32}{3}$.

2 Practice

Now apply the methods above.

2.1 Find where $y = x^2$ meets $y = 4$. [1]

2.2 Set up (do not evaluate) the integral for the area between $y = x^2$ and $y = 4$. [2]

2.3 Evaluate $\displaystyle\int_{0}^{2} (2x - x^2)\,dx$. [2]

3 Exam-style questions

3.1 The area between $y = f(x)$ (upper) and $y = g(x)$ (lower) on $[a, b]$ is [1]

- **A** $\displaystyle\int_a^b (g - f)\,dx$
- **B** $\displaystyle\int_a^b (f - g)\,dx$
- **C** $\displaystyle\int_a^b f\,dx$
- **D** $\displaystyle\int_a^b (f + g)\,dx$

3.2 Find the area of the region enclosed by $y = x$ and $y = x^2$.

(a) Find the limits of integration. [1]

(b) Evaluate the area. [3]

3.3 Find the area of the region enclosed by $y = 6x - x^2$ and $y = x$. [5]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **8.4 Area Between Curves (Functions of x)** lesson on the **Learn** page;
- read the **Finding the Area Between Curves Expressed as Functions of x** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $x^2 = 4 \Rightarrow x = \pm 2$.

2.2 $\displaystyle\int_{-2}^{2} (4 - x^2)\,dx$.

2.3 $\left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.

3.1 B —integrate top minus bottom.

3.2 (a) $x = 0, 1$. (b) $\displaystyle\int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$.

3.3 Intersect: $6x - x^2 = x \Rightarrow x^2 - 5x = 0 \Rightarrow x = 0, 5$; top is $6x - x^2$; $\displaystyle\int_0^5 \big((6x - x^2) - x\big)\,dx = \int_0^5 (5x - x^2)\,dx = \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = \frac{125}{2} - \frac{125}{3} = \frac{125}{6} \approx 20.8$.