AP CALCULUS BC • EXERCISE SHEET

8.13 The Arc Length of a Smooth Curve and Distance Traveled

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS arc length 弧长

Objective

Build the skills to answer exam questions on **arc length** — the length of a smooth curve.

You must be able to:

- apply $L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$
- set up the integral from a given function
- recognise the same idea as **distance travelled** along a path

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

The arc-length formula

The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

It adds up tiny hypotenuses $\sqrt{dx^2 + dy^2}$ along the curve.

Setting up the integral

For $y = \frac{2}{3}x^{3/2}$: $f'(x) = x^{1/2}$, so $\left(f'\right)^2 = x$, and

$$L = \int_0^3 \sqrt{1 + x}\, dx.$$

Evaluating

$$\int_0^3 \sqrt{1 + x}\, dx = \left[\tfrac{2}{3}(1 + x)^{3/2}\right]_0^3 = \tfrac{2}{3}\left(8 - 1\right) = \tfrac{14}{3}.$$

Distance along a path

If a particle moves along $y = f(x)$, the **distance travelled** is the same arc-length integral. Many arc-length integrals must be evaluated numerically — set them up correctly even when you cannot integrate by hand.

2 Practice

Now apply the methods above.

2.1 State the arc-length formula for $y = f(x)$ on $[a, b]$. [1]

2.2 For $y = \frac{2}{3}x^{3/2}$, find $f'(x)$ and $\left(f'(x)\right)^2$. [2]

2.3 Set up (do not evaluate) the arc length of $y = x^2$ from $x = 0$ to $x = 2$. [2]

3 Exam-style questions

3.1 The arc length of $y = f(x)$ on $[a, b]$ is [1]

- **A** $\int_a^b f(x)\, dx$
- **B** $\int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$
- **C** $\int_a^b f'(x)\, dx$
- **D** $\pi \int_a^b \left(f(x)\right)^2\, dx$

3.2 A curve is $y = \frac{2}{3}x^{3/2}$ for $0 \le x \le 3$.

(a) Set up the arc-length integral. [2]

(b) Evaluate it. [3]

3.3 Set up the integral for the length of $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$. (Use $\frac{d}{dx}\ln(\cos x) = -\tan x$ and $1 + \tan^2 x = \sec^2 x$.) [3]

Solutions

2.1 $L = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\,dx$.

2.2 $f'(x) = x^{1/2}$; $\big(f'(x)\big)^2 = x$.

2.3 $f'(x) = 2x$, so $L = \int_0^2 \sqrt{1 + 4x^2}\,dx$.

3.1 **B** — arc length integrates $\sqrt{1 + (f')^2}$.

3.2 (a) $f'(x) = x^{1/2}$, so $L = \int_0^3 \sqrt{1 + x}\,dx$. (b) $\left[\frac{2}{3}(1 + x)^{3/2}\right]_0^3 = \frac{2}{3}(8 - 1) = \frac{14}{3}$.

3.3 $f'(x) = -\tan x$, so $1 + (f')^2 = 1 + \tan^2 x = \sec^2 x$; $L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx$.