AP CALCULUS BC • EXERCISE SHEET

8.12 Volume with Washer Method: Revolving Around Other Axes

Name: ____________ Class: _________ Date: __________ **Total: 13 marks**

KEY WORDS washer method 垫圈法

Objective

Build the skills to answer exam questions on the **washer method around other axes** — a hole plus a shifted axis.

You must be able to:

- shift **both** radii to the distance from each curve to a line $y = k$ or $x = k$
- integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$ with the shifted radii
- sketch the region and label the two radii

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

Both radii shift

Revolving about a line $y = k$ moves **both** radii. Each becomes the distance from its curve to that line:

$$R_{\text{outer}} = |f_{\text{far}} - k|, \quad R_{\text{inner}} = |f_{\text{near}} - k|.$$

A worked example

Revolve the region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ about the line $y = -1$. Distances to $y = -1$: outer (from $\sqrt{x}$) $= \sqrt{x} + 1$; inner (from x) $= x + 1$:

$$V = \pi \int_0^1 \left((\sqrt{x} + 1)^2 - (x + 1)^2\right) dx.$$

Keep the order

The outer curve (farther from the axis) still gives the larger radius. When the axis is **below** the region, adding the same amount to each keeps their order.

Method summary

Sketch the region and the axis, mark R_{outer} and R_{inner} as distances to the axis, then integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$.

2 Practice

Now apply the methods above.

2.1 A curve $y = f(x)$ is revolved about $y = -2$. Write its radius. [1]

2.2 For outer curve $y = 4$ and inner curve $y = 1$ revolved about $y = -1$, state both radii. [2]

2.3 Expand $(x + 1)^2 - (\sqrt{x} + 1)^2$? No — first expand $(x + 1)^2$. [1]

3 Exam-style questions

3.1 Revolving a region between two curves about $y = k$ uses radii [1]

- **A** the curves themselves
- **B** each curve's distance to the line $y = k$
- **C** only the outer curve
- **D** k for both

3.2 The region between $y = x^2$ (lower) and $y = 2x$ (upper) on $[0, 2]$ is revolved about the line $y = -1$.

(a) Write the outer and inner radii. [2]

(b) Set up the volume integral. [2]

3.3 The region between $y = \sqrt{x}$ and $y = x$ on $[0, 1]$ is revolved about the line $y = 2$. Write the outer and inner radii (as distances to $y = 2$), and set up the volume integral. [4]

Solutions

2.1 $R = f(x) + 2$.

2.2 outer $= 4 - (-1) = 5$; inner $= 1 - (-1) = 2$.

2.3 $(x + 1)^2 = x^2 + 2x + 1$.

3.1 B — each radius is the curve's distance to the line $y = k$.

3.2 (a) On $(0, 2)$, $2x \geq x^2$, so outer (from $2x$) $= 2x + 1$, inner (from x^2) $= x^2 + 1$. (b) $V = \pi \int_0^2 \left((2x + 1)^2 - (x^2 + 1)^2\right) dx$.

3.3 Distances to $y = 2$: outer (from $y = x$, the lower curve, farther below 2) $= 2 - x$; inner (from $y = \sqrt{x}$) $= 2 - \sqrt{x}$; $V = \pi \int_0^1 \left((2 - x)^2 - (2 - \sqrt{x})^2\right) dx$.