AP CALCULUS BC • EXERCISE SHEET

8.11 Volume with Washer Method: Revolving Around the x- or y-Axis

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS washer method 垫圈法

Objective

Build the skills to answer exam questions on the **washer method** — volumes of revolution with a hole.

You must be able to:

- identify the **outer** and **inner** radii of a washer (ring)
- integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$ (the **washer method 垫圈法**)
- set each radius as a distance from a curve to the axis

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ When there is a hole

If the region does **not** touch the axis, revolving it leaves a hole, so each slice is a **washer** (a ring). Its area is (outer disc) − (inner disc):

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx.$$

■ Identifying the radii

Revolve the region between $y = \sqrt{x}$ (top) and $y = x$ (bottom) on $[0, 1]$ about the x-axis. Outer radius (further curve) $R = \sqrt{x}$; inner radius $r = x$:

$$V = \pi \int_0^1 \left((\sqrt{x})^2 - x^2\right) dx = \pi \int_0^1 (x - x^2)\, dx = \pi\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{\pi}{6}.$$

■ Do not square then subtract wrongly

$R^2 - r^2$ is **not** $(R - r)^2$. Square each radius separately, then subtract.

■ Choosing outer vs inner

The **outer** radius comes from the curve **farther** from the axis; the **inner** from the curve **nearer** the axis. A quick sketch settles which is which.

2 Practice

Now apply the methods above.

2.1 State the washer-method formula for the volume. [1]

2.2 For $R = 4$ and $r = 2$, find $\pi(R^2 - r^2)$. [1]

2.3 Evaluate $\pi \displaystyle\int_0^1 (x - x^2)\, dx$. [2]

3 Exam-style questions

3.1 The washer method integrates [1]

- **A** $\pi \displaystyle\int (R - r)^2\, dx$
- **B** $\pi \displaystyle\int (R^2 - r^2)\, dx$
- **C** $\pi \displaystyle\int R^2\, dx$
- **D** $\displaystyle\int (R^2 - r^2)\, dx$

3.2 The region between $y = x$ and $y = x^2$ (from $x = 0$ to $x = 1$) is revolved about the x-axis.

(a) Identify the outer and inner radii. [2]

(b) Set up and evaluate the volume. [4]

3.3 The region bounded by $y = 4$ and $y = x^2$ (for $0 \le x \le 2$) is revolved about the x-axis. Set up the washer integral for the volume. [3]

Solutions

2.1 $V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx$.

2.2 $\pi(16 - 4) = 12\pi$.

2.3 $\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$.

3.1 B — subtract the **squares** of the radii.

3.2 (a) On $(0, 1)$, $y = x$ is above $y = x^2$; outer $R = x$, inner $r = x^2$. (b) $V = \pi \int_0^1 (x^2 - x^4)\, dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$.

3.3 Outer $R = 4$, inner $r = x^2$; $V = \pi \int_0^2 \left(4^2 - (x^2)^2\right) dx = \pi \int_0^2 (16 - x^4)\, dx$.