AP CALCULUS BC • EXERCISE SHEET

7.8 Exponential Models with Differential Equations

Name: ______________ Class: __________ Date: ____________ **Total: 17 marks**

Objective

Build the skills to answer exam questions on **exponential models** —growth and decay from $\dfrac{dy}{dt} = ky$.

You must be able to:

- write the solution $y = y_0 e^{kt}$ from the model $\dfrac{dy}{dt} = ky$
- find k and y_0 from given data
- use the model to predict values and **half-life** / doubling situations

1 Worked examples

Study these first. Each one shows the method for a question type used later —follow the steps and you can do the Practice and Exam-style questions yourself.

■ The exponential solution

Any quantity whose rate of change is proportional to its size, $\dfrac{dy}{dt} = ky$, grows or decays exponentially:

$$y = y_0 e^{kt},$$

where $y_0 = y(0)$ is the starting amount. $k > 0$ is growth; $k < 0$ is decay.

■ Finding y0 and k from data

A culture starts at 500 and is 2000 after 3 hours. Then $y_0 = 500$ and $2000 = 500e^{3k}$, so $e^{3k} = 4$, $k = \frac{1}{3}\ln 4 \approx 0.462$. Model: $y = 500e^{0.462t}$.

■ Decay and half-life

For decay, $k < 0$. A substance with $y = y_0 e^{-0.1t}$ reaches **half** when $e^{-0.1t} = \frac{1}{2}$, so $-0.1t = \ln\frac{1}{2}$, $t = \dfrac{\ln 2}{0.1} \approx 6.93$ (the **half-life**).

■ Predicting a future value

With $y = 500e^{0.462t}$, the amount at $t = 5$ is $500e^{0.462(5)} = 500e^{2.31} \approx 5030$.

2 Practice

Now apply the methods above.

2.1 Write the solution of $\frac{dy}{dt} = 0.2y$ with $y(0) = 30$. [1]

2.2 A sample decays as $y = 80e^{-0.05t}$. Find the amount at $t = 10$. [2]

2.3 A population doubles: $y = y_0 e^{kt}$ with $y_0 = 100$ and $y(2) = 200$. Find k. [3]

3 Exam-style questions

3.1 The solution of $\frac{dy}{dt} = ky$ with $y(0) = y_0$ is [1]

- **A** $y = y_0 + kt$
- **B** $y = y_0 e^{kt}$
- **C** $y = kt^2$
- **D** $y = y_0 k^t$

3.2 A radioactive sample follows $\frac{dA}{dt} = kA$, starting at $A(0) = 40$ mg, and 30 mg remains after 5 years.

(a) Find k. [3]

(b) Predict the amount after 12 years. [2]

3.3 A bacteria count grows exponentially from 1000 to 8000 in 6 hours.

(a) Find the growth constant k. [2]

(b) How long, from the start, until the count reaches 16 000? [3]

4 Go further

You are now ready for the real exam questions on this subtopic:

- work through the **7.8 Exponential Models** lesson on the **Learn** page;
- read the **Exponential Models with Differential Equations** section of the AP Calculus BC handout on the **Know** page.

Solutions

2.1 $y = 30e^{0.2t}$.

2.2 $80e^{-0.5} = 80(0.6065) \approx 48.5$.

2.3 $200 = 100e^{2k}$; $e^{2k} = 2$; $k = \frac{1}{2}\ln 2 \approx 0.347$.

3.1 **B** —$\dfrac{dy}{dt} = ky$ gives $y = y_0 e^{kt}$.

3.2 (a) $30 = 40e^{5k}$; $e^{5k} = 0.75$; $k = \frac{1}{5}\ln 0.75 \approx -0.0575$. (b) $A = 40e^{-0.0575(12)} = 40e^{-0.690} \approx 20.0$ mg.

3.3 (a) $8000 = 1000e^{6k} \Rightarrow e^{6k} = 8 \Rightarrow k = \frac{1}{6}\ln 8 \approx 0.347$. (b) $16000 = 1000e^{kt} \Rightarrow e^{kt} = 16 \Rightarrow t = \dfrac{\ln 16}{0.347} \approx 8$ h.