

7.6 Finding General Solutions Using Separation of Variables

Name: _____ Class: _____ Date: _____

Total: 16 marks

KEY WORDS separable 可分离 • differential equation 微分方程

Objective

Build the skills to answer exam questions on **separation of variables** — the main method for solving a differential equation.

You must be able to:

- separate a **separable** 可分离 equation so each variable is on its own side
- integrate both sides and include a single constant C
- solve for y where the question asks for an explicit solution

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Separating the variables

For $\frac{dy}{dx} = xy$: put all y with dy and all x with dx :

$$\frac{1}{y} dy = x dx.$$

■ Integrating both sides

$$\int \frac{1}{y} dy = \int x dx \Rightarrow \ln |y| = \frac{x^2}{2} + C.$$

One constant C is enough (combine the two).

■ Solving for y

Exponentiate: $|y| = e^{x^2/2+C} = e^C e^{x^2/2}$. Writing $A = \pm e^C$ (a new constant),

$$y = Ae^{x^2/2}.$$

■ A quick separable example

$$\frac{dy}{dx} = \frac{x}{y}: y dy = x dx, \text{ so } \frac{1}{2}y^2 = \frac{1}{2}x^2 + C, \text{ i.e. } y^2 = x^2 + C_1.$$

2 Practice

Now apply the methods above.

2.1 Separate the variables in $\frac{dy}{dx} = 2xy$. [1]

2.2 Solve $\frac{dy}{dx} = \frac{1}{y}$ (find the general solution). [3]

2.3 Solve $\frac{dy}{dx} = 3y$ (general solution in terms of a constant A). [3]

3 Exam-style questions

3.1 After separating $\frac{dy}{dx} = ky$, integrating the left side gives [1]

- **A** y
- **B** $\ln |y|$
- **C** $\frac{1}{y^2}$
- **D** e^y

3.2 Solve $\frac{dy}{dx} = xy^2$.

(a) Separate and integrate both sides.

[3]

(b) Solve for y in terms of x and a constant.

[2]

3.3 Solve $\frac{dy}{dx} = \frac{\cos x}{y}$, giving the general solution.

[3]

Solutions

2.1 $\frac{1}{y} dy = 2x dx$.

2.2 $y dy = dx$; $\frac{1}{2}y^2 = x + C$; $y^2 = 2x + C_1$.

2.3 $\frac{1}{y} dy = 3 dx$; $\ln |y| = 3x + C$; $y = Ae^{3x}$.

3.1 B — $\int \frac{1}{y} dy = \ln |y|$.

3.2 (a) $\frac{1}{y^2} dy = x dx$; $-\frac{1}{y} = \frac{x^2}{2} + C$. (b) $y = -\frac{1}{\frac{x^2}{2} + C} = \frac{-2}{x^2 + C_1}$.

3.3 $y dy = \cos x dx$; $\frac{1}{2}y^2 = \sin x + C$; $y^2 = 2 \sin x + C_1$.