

6.9 Integrating Using Substitution

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS integral 积分 • definite integral 定积分

Objective

Build the skills to answer exam questions on **integration by substitution** (u -substitution) — reversing the chain rule.

You must be able to:

- choose $u = g(x)$ so that its derivative $g'(x)$ also appears in the integrand
- convert dx to du and integrate in terms of u
- handle a **definite** integral by changing the limits or converting back to x

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Choosing u

Look for an “inside function” whose derivative is also present. For $\int 2x(x^2 + 1)^3 dx$, let $u = x^2 + 1$, so $du = 2x dx$. The integral becomes

$$\int u^3 du = \frac{u^4}{4} + C = \frac{(x^2 + 1)^4}{4} + C.$$

■ Adjusting for a constant

If the derivative is off by a constant, fix it. For $\int x(x^2 + 1)^3 dx$, let $u = x^2 + 1$, $du = 2x dx$, so $x dx = \frac{1}{2} du$:

$$\int u^3 \cdot \frac{1}{2} du = \frac{u^4}{8} + C.$$

■ Definite integral — change the limits

For $\int_0^2 2x(x^2 + 1)^3 dx$, with $u = x^2 + 1$: when $x = 0$, $u = 1$; when $x = 2$, $u = 5$. So

$$\int_1^5 u^3 du = \left[\frac{u^4}{4} \right]_1^5 = \frac{625 - 1}{4} = 156.$$

■ A trig example

$\int \cos(3x) dx$: let $u = 3x$, $du = 3 dx$, so $dx = \frac{1}{3} du$, giving $\frac{1}{3} \sin(3x) + C$.

2 Practice

Now apply the methods above.

2.1 For $\int 3x^2(x^3 + 5)^4 dx$, state a good choice of u and find du . [2]

2.2 Find $\int 2x e^{x^2} dx$. [2]

2.3 Find $\int \sin(4x) dx$. [2]

3 Exam-style questions

3.1 To integrate $\int x \cos(x^2) dx$, the best substitution is [1]

- A $u = x$
- B $u = \cos x$
- C $u = x^2$
- D $u = \cos(x^2)$

3.2 Find $\int (2x + 1)(x^2 + x)^5 dx$. [3]

3.3 Evaluate $\int_0^1 3x^2\sqrt{x^3 + 1} dx$ using a substitution, changing the limits. [4]

Solutions

2.1 $u = x^3 + 5$; $du = 3x^2 dx$.

2.2 $u = x^2$, $du = 2x dx$; $\int e^u du = e^{x^2} + C$.

2.3 $u = 4x$, $dx = \frac{1}{4}du$; $-\frac{1}{4}\cos(4x) + C$.

3.1 C — $u = x^2$ gives $du = 2x dx$, matching the x in the integrand.

3.2 $u = x^2 + x$, $du = (2x + 1) dx$; $\int u^5 du = \frac{u^6}{6} = \frac{(x^2 + x)^6}{6} + C$.

3.3 $u = x^3 + 1$, $du = 3x^2 dx$; limits $x = 0 \rightarrow u = 1$, $x = 1 \rightarrow u = 2$; $\int_1^2 \sqrt{u} du = \left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{2}{3}(2\sqrt{2} - 1) \approx 1.22$.