

6.13 Evaluating Improper Integrals

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS diverges 发散 • improper integral 反常积分 • limit 极限 • converges 收敛
• integral 积分

Objective

Build the skills to answer exam questions on **improper integrals** — integrals with an infinite limit or an infinite discontinuity.

You must be able to:

- rewrite an improper integral as a **limit** 极限
- evaluate the limit to decide **convergence** 收敛 or **divergence** 发散
- handle a discontinuity inside the interval

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Infinite upper limit

Replace ∞ with b and take a limit:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1.$$

The limit is finite, so the integral **converges** to 1.

■ A divergent example

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln x]_1^b = \lim_{b \rightarrow \infty} \ln b = \infty. \text{ The limit is infinite, so it } \mathbf{diverges}.$$

■ Discontinuity at an endpoint

$$\int_0^1 \frac{1}{\sqrt{x}} dx \text{ is improper at } x = 0: \lim_{a \rightarrow 0^+} \int_a^1 x^{-1/2} dx = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^1 = 2 - 0 = 2. \\ \text{Converges.}$$

■ The rule for $1/x^p$ on $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ } \mathbf{converges} \text{ when } p > 1 \text{ and } \mathbf{diverges} \text{ when } p \leq 1 \text{ — a handy check.}$$

2 Practice

Now apply the methods above.

2.1 Rewrite $\int_1^{\infty} \frac{1}{x^3} dx$ as a limit. [1]

2.2 Evaluate $\int_1^{\infty} \frac{1}{x^3} dx$. [3]

2.3 State whether $\int_1^{\infty} \frac{1}{x^{1/2}} dx$ converges or diverges (use the p -rule). [1]

3 Exam-style questions

3.1 $\int_1^{\infty} \frac{1}{x^p} dx$ converges when [1]

- **A** $p < 1$
- **B** $p = 1$
- **C** $p > 1$
- **D** all p

3.2 Evaluate $\int_0^{\infty} e^{-x} dx$.

(a) Write it as a limit. [1]

(b) Evaluate. [3]

3.3 Determine whether $\int_2^\infty \frac{1}{x} dx$ converges or diverges, showing your reasoning with a limit. [3]

Solutions

2.1 $\lim_{b \rightarrow \infty} \int_1^b x^{-3} dx.$

2.2 $\left[-\frac{1}{2x^2}\right]_1^b = -\frac{1}{2b^2} + \frac{1}{2};$ as $b \rightarrow \infty$ this $\rightarrow \frac{1}{2}.$

2.3 Diverges — $p = \frac{1}{2} \leq 1.$

3.1 C — converges for $p > 1.$

3.2 (a) $\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx.$ (b) $\left[-e^{-x}\right]_0^b = -e^{-b} + 1; \rightarrow 1$ as $b \rightarrow \infty.$

3.3 $\lim_{b \rightarrow \infty} [\ln x]_2^b = \lim_{b \rightarrow \infty} (\ln b - \ln 2); \ln b \rightarrow \infty,$ so it **diverges**.